Financial Mathematics, 640:495: Black-Scholes Pricing

1. Random Process Models for Asset Prices.

We formalize the definition of a random process model for an asset price as follows. Let $\{S_t;\, t \geq 0\}$ denote the price of a risky asset. A random process model for this price is a characterization of the random nature of the price process that determines, in principle, the probability $I\!P(A)$ of any set A of price history outcomes, and the expectation $E[Z]$ of any random variable Z determined by the price history. For example, a properly formulated model will determine probabilities of the sort

$$I\!P\left(a_1 < S_{t_1} \leq b_1, \ldots, a_n < S_{t_n} \leq b_n\right),$$

for any set of times $0 \leq t_1 < t_2 < \cdots < t_n$ and intervals $(a_1, b_1], \ldots, (a_n, b_n]$; or probabilities such as

$$I\!P\left(\max_{[0,T]} S_t > a\right), \quad \text{for any } T \text{ and } a;$$

and also expectations such as $E\left[C(S_T)\right]$ or $E\left[\max_{[0,T]} S_t\right]$. These are just examples; the point is the models specifies probabilities of any event depending on $\{S_t;\, t \geq 0\}$.

Example. The Black-Scholes price model introduced in previous lectures is

$$S_t = S_0 \exp\{\mu t + \sigma B_t - \frac{\sigma^2}{2} t\},$$

where B represents a standard Brownian motion. This model shall be denoted $BS(\mu, \sigma^2)$. The definition of Brownian motion fully determines how to compute probabilities of any events concerning the price process, or the expectations of any values depending on the price process. We give some examples as practice in working with Brownian motion and using its properties.

1. Consider a call option on $\{S_t;\, t \geq 0\}$ at expiration T and strike X. What is the probability that the option expires in the money?

Solution: The problem asks for $I\!\!P\,(S_T > X)$. This is

$$\begin{aligned} I\!\!P\left((\mu-\frac{\sigma^2}{2})T+\sigma B_T > \ln(X/S_0)\right) &= I\!\!P\left(\frac{B_T}{\sigma\sqrt{T}} > \frac{\ln(X/S_0)+(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}}\right) \\ &= 1 - N\left(\frac{\ln(X/S_0)+(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}}\right) \\ &= N\left(-\frac{\ln(X/S_0)+(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}}\right) \\ &= N\left(\frac{\ln(S_0/X)+(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \end{aligned}$$

since $B_T/(\sigma\sqrt{T})$ is an $N(0,1)$ random variable.

2. Find the expected pay-off of the call option of the previous example, but at a general expiration date T.

   Solution: This problem asks for $E\left[(S_T - X)^+\right]$.

   Note that, as in problem 1, $S_T > X$ if and only if

$$\sigma B_T > \ln(X/S_0)+(\frac{\sigma^2}{2}-\mu)T,$$

   which occurs if and only if

$$\frac{B_T}{\sqrt{T}} > \frac{\ln(X/S_0)+(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}}. \qquad (1)$$

   Let $Z = B_T/\sqrt{T}$, and let

$$d_2 = -\frac{\ln(X/S_0)+(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}} = \frac{\ln(S_0/X)+(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

   Hence (1) is the same as $Z > -d_2$.

   Notice that $S_T = S_0\exp\{(\mu-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\}$. Therefore, because Z is $N(0,1)$

$$E\left[(S_T - X)^+\right] = E\left[\left(S_0 e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z} - X\right)^+\right]$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left(S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}z} - X \right)^+ \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} \\
&= \int_{-d_2}^{\infty} \left(S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z} - X \right) \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} \\
&= S_0 e^{\mu T} \int_{-d_2}^{\infty} e^{-(z-\sigma\sqrt{t})^2/2} \frac{dz}{\sqrt{2\pi}} - X \int_{-d_2}^{\infty} \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}}
\end{aligned}
$$

Now,

$$
\int_{-d_2}^{\infty} \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} = 1 - N(-d_2) = N(d_2).
$$

Also, by the change of variable $w = z + \sigma\sqrt{T}$,

$$
\int_{-d_2}^{\infty} e^{-(z-\sigma\sqrt{t})^2/2} \frac{dz}{\sqrt{2\pi}} = \int_{-d_2-\sigma\sqrt{T}}^{\infty} \frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} = N(d_2 + \sigma\sqrt{T}).
$$

In conclusion,

$$
E\left[(S_T - X)^+\right] = S_0 e^{\mu T} N(d_1) - X N(d_2) \tag{2}
$$

where

$$
d_2 = \frac{\ln(S_0/X) + (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_1 = d_2 + \sigma\sqrt{T} = \frac{\ln(S_0/X) + (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \tag{3}
$$

3. We stated in a previous lecture that with probability 1,

$$
\lim_{n\to\infty} \sum_{1}^{n} \left(B_{kt/2^n} - B_{(k-1)t/2^n} \right)^2 = t
$$

with probability 1. We write this as

$$
\mathbb{P}\,(\text{quadratic variation of } B \text{ on } [0,t] \text{ equals } t) = 1 \tag{4}
$$

Assuming the Samuelson model $BS(\mu, \sigma)$, $\ln S_t = \ln S_0 + \sigma B_t + (\mu - (\sigma^2/2))t$ and one can thus derive

$$
\mathbb{P}\,(\text{quadratic variation of } \ln S \text{ on } [0,t] \text{ equals } \sigma^2 t) = 1 \tag{5}
$$

2. Equivalent models; Black-Scholes pricing framework. Suppose Alice has proposed model, Model I, for an asset price, and let $I\!P_1(A)$ denote probability of an event A computed under her model. Suppose Bob proposes a different model, Model II, for the same asset price, and let probabilities for this model be denoted $I\!P_2(A)$. Model I and Model II are said to be equivalent, written $I\!P_1 \sim I\!P_2$ if

$$I\!P_1(A) > 0 \quad \text{if and only if} \quad I\!P_2(A) > 0. \tag{6}$$

In words, Alice and Bob have equivalent models if they agree on which events have positive probability and which have zero probability.

In general, if Alice and Bob do not believe in equivalent models they will not be able to agree on a fair (no-arbitrage) price of an option. Alice, in determining what price she would be willing to buy or sell a derivative contract, will ask what amount of money she will need to hedge the position. This will depend on which outcomes she deems have positive probability. If Bob has a different idea of what outcomes have positive probability, the amount he thinks he needs to hedge may be different.

Example. The Black-Sholes models $BS(\mu, \sigma_1^2)$ and $BS(\mu, \sigma_2)$ are not equivalent unless $\sigma_1^2 = \sigma_2^2$. This follows because of the third example in section 1 of this lecture. Under $BS(\mu, \sigma_1^2)$

$$I\!P\,(\text{quadratic variation of } \ln S \text{ on } [0,t] \text{ equals } \sigma_1^2 t) = 1, \tag{7}$$

but under $BS(\mu, \sigma_2^2)$ this event has probability zero because the quadratic variation over $[0,t]$ is $\sigma_2^2 t$ instead. ◇

Black-Scholes models with different volatilities are not equivalent, but for drifts we have the following result.

Theorem 1 *For any drifts μ_1 and μ_2 and a single volatility σ^2, $BS(\mu_1, \sigma^2)$ and $BS(\mu_2, \sigma^2)$ are equivalent.*

This theorem is beyond our means to prove; in fact it relates to very deep facts about Brownian motion. But perhaps it should not be surprising, because a simple, deterministic transformation turns a $BS(\mu_1, \sigma_2)$ model into a $BS(\mu_2, \sigma^2)$ model. Indeed, if $S_t = S_0 \exp\{(\mu_1 - (\sigma^2/2))t + \sigma B_t\}$ follows a $BS(\mu_1, \sigma^2)$ model, then $e^{(\mu_2 - \mu_1)t} S_t = S_0 \exp\{(\mu_2 - (\sigma^2/2))t + \sigma B_t\}$ follows a $BS(\mu_2, \sigma^2)$ model.

Definition. We will say that we are *pricing in the Black-Scholes framework* or *doing Black-Scholes pricing* if we assume a Black-Scholes model $BS(\mu, \sigma^2)$ as a *reference model* for the asset price. This means that all market participants have models equivalent to $BS(\mu, \sigma^2)$, that is, they agree with $BS(\mu, \sigma^2)$ on which events have probability zero and which have positive probability.

3. Risk-neutral pricing strategy. The pricing philosophy we use generalizes pricing for the binomial tree model. Suppose we are given a risk-free interest rate and a reference model for an asset price $\{S_t;\, t \geq 0\}$, assigning probabilities $I\!P(A)$ to market history events A. We seek a model $\tilde{I\!P}$ satisfying: (a) $\tilde{I\!P}$ is equivalent to $I\!P$, and (b), assuming model $\tilde{I\!P}$ the discounted price process $\{e^{-rt}S_t;\, t \geq 0\}$ is a martingale (relative to its own past). The measure $\tilde{I\!P}$ in this context is called the *equivalent risk-neutral measure*. Then, the price V_0 of a derivative with random pay-off V_T at time T is

$$V_0 = e^{-rT}\tilde{E}\left[V_T\right], \tag{8}$$

where $\tilde{E}$ represents expectation using $\tilde{p}$. More generally, the price at time t, $0 < t < T$, is the conditional expectation,

$$V_t = e^{-r(T-t)}\tilde{E}\left[V_T \mid S_u,\, u \leq t\right], \tag{9}$$

For this formula to give us an unambiguous answer, we would like to have a situation in which the equivalent risk-neutral measure is unique.

In the binomial tree model the formula (9) was arrived at by no-arbitrage pricing. In hindsight the reasoning can be summarized as follows. Given a trading strategy $\triangle$ and an initial endowment P_0, let Π_t be the value at time t of the portfolio which starts with P_0 and uses strategy $\triangle$. The discounted value $e^{-rt}\Pi_t$ can make a gain or a profit only by trading on the fluctuations of the discounted asset price $e^{-rt}S_t$; this is intuitively clear because by discounted we remove any gains from investments at the risk free rate. The profit from trading on the fluctuations of a martingale is still a martingale; this is the principle that one cannot by trading turn a fair game into a favorable or unfavorable game. Therefore, $e^{-rt}\Pi_t$ must be a martingale assuming the risk-neutral equivalent model. This implies

$$e^{-rt}\Pi_t = \tilde{E}\left[e^{-rT}\Pi_T \mid S_u,\, u \leq t\right],$$

or equivalently

$$\Pi_t = e^{-r(T-t)}\tilde{E}\left[\Pi_T \mid S_u,\, u \leq t\right] \tag{10}$$

Now suppose that we have a trading strategy so that $\Pi_T = V_T$, that is, that duplicates the derivative payoff. Then Π_t is the capital one needs at time t to duplicate V_T and so, if there is to be no arbitrage the price of the derivative at time t should be $V_t = \Pi_t$. Substituting $V_t = \Pi_t$ and $V_T = \Pi_T$ into (10) gives the pricing formula (9).

4. Black-Scholes pricing and the Black-Scholes formula for a call option.

Now we are ready to set up pricing in the Black-Scholes model. Recall from a previous lecture that if B is a standard Brownian motion then

$$\exp\{\sigma B_t - \frac{\sigma^2}{2}t\} \quad \text{is a martingale.} \tag{11}$$

Now suppose the reference model for an underlying asset is the Black-Scholes model $BS(\mu, \sigma^2)$, where μ and σ^2 are given. Let r be the risk-free interest rate, as usual. The initial price S_0 is given an known also. We claim that $BS(r, \sigma^2)$ is an equivalent risk-neutral model. This is easy. We know it is equivalent by Theorem 1 stated above. It is risk-neutral because

$$e^{-rt}S_t = \exp\{-rt\}\exp\{(r - \frac{\sigma^2}{2})t + \sigma B_t\} = \exp\{\sigma B_t - \frac{\sigma^2}{2}t\},$$

and we know from (11) that this is a martingale. It is a **fact** that we cannot prove in this course, that $BS(r, \sigma^2)$ is the unique risk-neutral model equivalent to $BS(\mu, \sigma^2)$.

According to section 3 of this lecture, if V_T represents the payoff of any derivative written on the underlying asset, its price at time t is

$$V_t = e^{-r(T-t)}\tilde{E}\left[V_T \mid S_u,\, u \le t\right], \tag{12}$$

where $\tilde{I\!P}$ denotes expectation assuming S follows the $B(r, \sigma^2)$ model.

At time $t = 0$, there is nothing to condition on and V_0 is given by (8) using the $BS(r, \sigma^2)$ model. Thus

$$V_0 = e^{-rT}\tilde{E}\left[V_T\right], \tag{13}$$

Example. **The Black-Scholes formula for the price of a call option.** Consider a call option on an underlying asset with expiration T and strike

X. Assume the the Black-Scholes model with volatility σ^2 for the asset price process $\{S_t;\, t \geq 0\}$. Let r be the risk-free interest rate. The payoff of the option is $V_T = (S_t - X)^+$. Hence, the price of the call at time 0 is given by (13):

$$V_0 = \tilde{E}\big[\big(S_0 \exp\{(r - (\sigma^2/2))T + \sigma B_T\} - X\big)^+\big]\,.$$

But we did this above when r is instead a general μ and gave the answer in (2) and (3). Thus, replacing μ there by r, we derive the Black-Scholes formula for the price of a call option.

$$V_0 = S_0 e^{\mu T} N(d_1(r, \sigma^2, T)) - XN(d_2(r, \sigma^2, T)) \tag{14}$$

where

$$\begin{aligned} d_2(r, \sigma^2, T) &= \frac{\ln(S_0/X) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_1(r, \sigma^2, T) &= d_2(T) + \sigma\sqrt{T} = \frac{\ln(S_0/X) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \end{aligned} \tag{15}$$

What about the price at an intermediate time t. We use that

$$S_T = S_t \exp\{(r - (\sigma^2/2))(T - t) + \sigma(B_T - B_t)\},$$

and the familiar fact that the increment $B_T - B_t$ is **independent** of the price history $\{S_u; , u \leq t\}$ up to time t. Therefore, the problem of pricing an option at time t with current price S_t, when the option expires at T, is like the problem of pricing it from time 0, but with S_0 replaced by S_t and T, by the new time to expiration $T+t$. Hence we have

$$\begin{aligned} V_t &= e^{-r(T-t)} \tilde{E}\left[\left(S_t e^{(r - (\sigma^2/2))(T-t) + \sigma(B_T - B_t)} - X\right)^+ \mid S_u, u \leq t\right] \\ &= S_t N\left(d_1(r, \sigma^2, T-t)\right) - e^{-r(T-t)} XN\left(d_1(r, \sigma^2, T-t)\right). \end{aligned} \tag{16}$$

To see this more formally, note that, because $B_T - B_t$ is independent of $\{S_u; , u \leq t\}$, because $(B_T - B_t)/\sqrt{T-t}$ is $N(0,1)$, and because S_t is known if $\{S_u; , u \leq t\}$ is known, the familiar rules of calculating expectations give us,

$$V_t = \int_{-\infty}^{\infty} (S_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(T-t)z} - X)^+ \frac{e^{-z^2/2}\, dz}{\sqrt{2\pi}}.$$

Then we just apply the calculations leading to (refeq2) and (3), but with T replaced by $T-t$ and S_0 by S_t. This gives (16).