640:495 Mathematical Finance, Problems.

In class we discussed the fact that if $\alpha(Y_1,\ldots,Y_n)$ and $\beta(Y_1,\ldots,Y_n)$ are functions of $Y_1,\ldots,Y_n$, then, abbreviating them by α and β, we have the identity

$$E\left[\alpha+\beta X \mid Y_1,\ldots,Y_n\right]=\alpha+\beta E\left[X \mid Y_1,\ldots,Y_n\right]. \tag{1}$$

Te idea is that once $Y_1,\ldots,Y_n$ are known, then the values of α and β are known and behave as constants with respect to the conditional expectation.. This identity is applied in the online notes for lecture 15, in the section on martingales. For example, in the example on page 13 showing the symmetric random walk is a martingale, we use it to write

$$E\left[x+\sum_1^n Y_i+Y_{n+1} \mid Y_1,\ldots,Y_n\right]=x+\sum_1^n Y_i+E[Y_{n+1} \mid Y_1,\ldots,Y_n].$$

In the example on geometric random walk on page 14, it is used to conclude

$$E\left[\lambda^{Y_1+\cdots+Y_n}\lambda^{Y_{n+1}} \mid Y_1,\ldots,Y_n\right]=\lambda^{Y_1+\cdots+Y_n}E\left[\lambda^{Y_{n+1}} \mid Y_1,\ldots,Y_n\right]$$

This discussion may look very abstract to you. The next few problems will help you understand this as they are to apply formula (1) in more concrete, but similar instances. They will also prepare you for problems 50,51,and 52. Recall for these problems also the fact that if X is independent of $Y_1,\ldots,Y_n$, $\mathbb{P}\left(X=x \mid Y_1=y_1,\ldots,Y_n=y_n\right)=\mathbb{P}(X=x)$, and so $E[X \mid Y_1,\ldots,Y_n]=E[X]$.

53. Let Y_1,Y_2,Y_3 be independent random variables with $\mathbb{P}(Y_i=1)=1/3$ and $\mathbb{P}(Y_i=-1)=2/3$ for each i. Find

$$(a)\quad E\left[Y_1+Y_2+Y_3 \mid Y_1,Y_2\right] \qquad \text{and} \qquad E\left[Y_1Y_2Y_3 \mid Y_1,Y_2\right]$$

as explicit functions of Y_1 and Y_2. Do the same for

$$(c)\qquad E\left[e^{Y_1+Y_2+Y_3} \mid Y_1,Y_2\right].$$

If you can do this exercise, problem 50 should be easy.

54. This is a followup to problem 50. As in that problem, let $\xi_1,\xi_2,\ldots$ be independent random variables, each with the same distribution, $\xi_i=1$ with probability p and $\xi_i=-1$ with probability $q=1-p$. For each n, let $X_n=\sum_{i=1}^n \xi_i$. Show $Z_n=X_n-n(2p-1)=X_n-n(p-q)$ is a martingale (relative to $\{\xi_n\}$).

55. As in problems 50 and 54, let $\xi_1,\xi_2,\ldots$ be independent random variables, each with the same distribution, $\xi_i=1$ with probability p and $\xi_i=-1$ with probability $q=1-p$. For each n, let $X_n=\sum_{i=1}^n \xi_i$.

Show that $E\left[\left(\frac{q}{p}\right)^{\xi_i}\right] = 1.$

Show that $Z_n = \left(\frac{q}{p}\right)^{X_n}$ is a martingale (relative to $\{\xi_n\}$).

56. Let $\{X_n\}$ be a martingale relative to $\{Y_n\}$. Show that $E[X_n] = E[X_0]$ for all n. So martingales are constant in expectation. (Hint: In $E[X_n]$ replace X_n by $E[X_n \mid Y_1, \ldots, Y_{n-1}]$ and use the martingale property. Continue this procedure.)

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57. Consider the two period binomial model with $S_0 = 50$, and $g = 1.2$, $\ell = 0.8$ and $e^{r\tau} = 1.05$. At time $t = 0$ you are given \$300. You engage in the following trading strategy. At time $t = 0$ you buy $\triangle_0 = 4$ stocks and invest the rest at the risk-free rate. At time $t = 1$, you rebalance to have $\triangle_1(u) = 5$ stocks if the market goes up (u) in the first period, but $\triangle_1(d) = 3$ shares if the market goes down, always investing the rest at the risk-free rate.

Let $\Pi_t(w_1, \ldots, w_t)$ denote the value of your portfolio at time t if $w_1, \ldots, w_t$ is the market history. For the scenario above, $\Pi_0 = 300$ is given. Compute numerical values for $\Pi_1(u), \Pi_1(d), \Pi_2(u,u), \Pi_2(u,d), \Pi_2(d,u), \Pi_2(d,d)$.

In class we stated that under the risk-free measure on market histories, $\{e^{-tr\tau}\Pi_t\}$ is a martingale. In particular, from the previous problem, $\tilde{E}\left[e^{-2r\tau}\Pi_2\right] = \Pi_0 = 300$. Verify this numerically, by directly computing the expectation on the left-hand side. (Your answer may be off a small amount because of round-off error.)