A–1 Determine, with proof, the number of ordered triples 
\((A_1, A_2, A_3)\) of sets which have the property that 
(i) \(A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\) and 
(ii) \(A_1 \cap A_2 \cap A_3 = \emptyset\).

Express your answer in the form \(2^a3^b5^c7^d\), where \(a, b, c, d\) are nonnegative integers.

A–2 Let \(T\) be an acute triangle. Inscribe a rectangle \(R\) in \(T\) with one side along a side of \(T\). Then inscribe a rectangle \(S\) in the triangle formed by the side of \(R\) opposite the side on the boundary of \(T\), and the other two sides of \(T\), with one side along the side of \(R\). For any polygon \(X\), let \(A(X)\) denote the area of \(X\). Find the maximum value, or show that no maximum exists, of \(\frac{A(R) + A(S)}{A(T)}\), where \(T\) ranges over all triangles and \(R, S\) over all rectangles as above.

A–3 Let \(d\) be a real number. For each integer \(m \geq 0\), define a sequence \(\{a_m(j)\}, j = 0, 1, 2, \ldots\) by the condition
\[a_m(0) = \frac{d}{2^m}, \quad a_m(j + 1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.\]
Evaluate \(\lim_{n \to \infty} a_n(n)\).

A–4 Define a sequence \(\{a_i\}\) by \(a_1 = 3\) and \(a_{i+1} = 3a_i\) for \(i \geq 1\). Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many \(a_i\)?

A–5 Let \(I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx\). For which integers \(m, 1 \leq m \leq 10\) is \(I_m \neq 0\)?

A–6 If \(p(x) = a_0 + a_1x + \cdots + a_mx^m\) is a polynomial with real coefficients \(a_i\), then set
\[
\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.
\]
Let \(F(x) = 3x^2 + 7x + 2\). Find, with proof, a polynomial \(g(x)\) with real coefficients such that
(i) \(g(0) = 1\), and
(ii) \(\Gamma(f(x)^n) = \Gamma(g(x)^n)\)
for every integer \(n \geq 1\).

B–1 Let \(k\) be the smallest positive integer for which there exist distinct integers \(m_1, m_2, m_3, m_4, m_5\) such that the polynomial
\[p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)\]
has exactly \(k\) nonzero coefficients. Find, with proof, a set of integers \(m_1, m_2, m_3, m_4, m_5\) for which this minimum \(k\) is achieved.

B–2 Define polynomials \(f_n(x)\) for \(n \geq 0\) by \(f_0(x) = 1\), \(f_n(0) = 0\) for \(n \geq 1\), and
\[
\frac{d}{dx}f_{n+1}(x) = (n + 1)f_n(x + 1)
\]
for \(n \geq 0\). Find, with proof, the explicit factorization of \(f_{100}(1)\) into powers of distinct primes.

B–3 Let
\[
a_{1,1} \quad a_{1,2} \quad a_{1,3} \quad \cdots \quad a_{2,1} \quad a_{2,2} \quad a_{2,3} \quad \cdots \quad a_{3,1} \quad a_{3,2} \quad a_{3,3} \quad \cdots
\]
be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that \(a_{m,n} > mn\) for some pair of positive integers \((m,n)\).

B–4 Let \(C\) be the unit circle \(x^2 + y^2 = 1\). A point \(p\) is chosen randomly on the circumference \(C\) and another point \(q\) is chosen randomly from the interior of \(C\) (these points are chosen independently and uniformly over their domains). Let \(R\) be the rectangle with sides parallel to the \(x\) and \(y\)-axes with diagonal \(pq\). What is the probability that no point of \(R\) lies outside of \(C\)?

B–5 Evaluate \(\int_0^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\). You may assume that \(\int_0^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\).

B–6 Let \(G\) be a finite set of real \(n \times n\) matrices \(\{M_i\}\), \(1 \leq i \leq r\), which form a group under matrix multiplication. Suppose that \(\sum_{i=1}^r \text{tr}(M_i) = 0\), where \(\text{tr}(A)\) denotes the trace of the matrix \(A\). Prove that \(\sum_{i=1}^r M_i\) is the \(n \times n\) zero matrix.