Minima and Maxima

1. Over all sequences of positive integers that sum to 1000, determine the one whose product is maximum.

2. A collection of tennis players play a tournament in which each pair of players plays one match. A player \( p \) is called weakly dominant if for every other player \( q \), \( p \) beat \( q \) or \( p \) beat at least one player that beat \( q \). Prove that there is at least one weakly dominant player.

3. If \( S \) is a set of real numbers, let \( A(S) \) be the set of numbers of the form \((x + y)/2\) where \( x, y \in S \). For each positive integer \( n \) determine the minimum size of \( A(S) \) over all sets \( S \) of size \( n \).

4. Suppose that \( f \) is a continuous function on \([0, 1]\) such that for each \( j \in \{0, 1, \ldots, n - 1\} \)
\[
\int_0^1 x^j f(x) = 0.
\]
and
\[
\int_0^1 x^n f(x) = 1.
\]
Prove that the maximum of \( |f(x)| \) on \([0, 1]\) is greater than \( 2^n(n + 1) \).

5. Putnam 1998:B1) Find the minimum value of:
\[
\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}.
\]
for \( x > 0 \).

6. (Putnam 1993:B1) Find the smallest positive integer \( n \) such that for every integer \( m \) with \( 0 < m < 1993 \), there is an integer \( k \) for which \( m/1993 < k/n < (m + 1)/1993 \).

7. (Putnam 1988:B3) For every \( n \) in the set \( \mathbb{N} \) of positive integers, let \( r_n \) be the minimum of \( |c - d\sqrt{3}| \) over all nonnegative integers \( c \) and \( d \) with \( c + d = n \). Find, with proof, the lease positive real number \( y \) such that \( r_n \leq y \) for all \( n \in \mathbb{N} \).

8. Putnam 1991:B6. Let \( a, b \) be positive numbers. Find the largest number \( c \) in terms of \( a \) and \( b \) such that
\[
a^x b^{1-x} \leq a \frac{\sinh u x}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u},
\]
for all \( u \) satisfying \( 0 \leq |u| \leq c \) and for all \( x, 0 < x < 1 \).