Suggested problems for 10/31/06

1. Let \( P \) be a non-constant polynomial given by:

\[
P(x) = a_nx^n + \cdots + a_1x + a_0.
\]

Assume that \( P \) has \( n \) distinct nonzero roots \( r_1, \ldots, r_n \). Prove that:

\[
\frac{1}{r_1} + \cdots + \frac{1}{r_n} = -\frac{a_1}{a_0}.
\]

2. For \( k \) a nonnegative integer and \( x \) a variable define:

\[
\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}.
\]

Let \( P(x) \) be a real polynomial of degree \( n \).

(a) Prove that there are unique reals \( a_0, a_1, \ldots, a_n \) such that \( P(x) = a_0\binom{x}{0} + a_1\binom{x}{1} + \cdots a_n\binom{x}{n} \).

(b) Prove that the coefficients \( a_0, \ldots, a_n \) in the first part are all integers if and only if \( P \) maps integers to integers.

3. Prove that if the quadratics \( ax^2 + bx + c \) and \( px^2 + qx + r \) have a common root then \((ar - cp)^2 = (aq - bp)(br - cq)\).

4. Prove that if \( P \) and \( Q \) are polynomials with \( P^2 - Q^3 = 1 \) then \( P \) and \( Q \) are constant polynomials.

5. Let \( P(x) \) be a polynomial of degree \( n \geq 1 \) and distinct roots \( r_1, \ldots, r_n \). Prove that for any number \( a \) such that \( P'(a) \neq 0 \) there is an \( i \in \{1, \ldots, n\} \) such that \( |a - r_i| \leq n|P(a)|/|P'(a)| \).

6. (Putnam 1985, B2) Define polynomials \( f_n(x) \) for \( n \geq 0 \) by \( f_0(x) = 1 \), \( f_n(0) = 0 \) for \( n \geq 1 \) and \( f'_{n+1}(x) = (n+1)f_n(x+1) \) for \( n \geq 0 \). Find with proof, the explicit factorization of \( f_{100}(1) \) into powers of distinct primes.

7. (Putnam 1986, A6) Let \( a_1, \ldots, a_n \) be real numbers and \( b_1, \ldots, b_n \) be distinct positive integers. Suppose there is a polynomial \( f(x) \) satisfying:

\[
(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.
\]

Find a simple expression (not involving summations) for \( f(1) \) in terms of \( b_1, \ldots, b_n \) and \( n \) (but independent of \( a_1, \ldots, a_n \)).

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8. (Putnam 1986, B5). Let \( f(x, y, z) = x^2 + y^2 + z^2 + xyz \). Let \( p(x, y, z), q(x, y, z), r(x, y, z) \) be real polynomials satisfying:

\[
 f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).
\]

Prove or disprove the assertion that the sequence \( p, q, r \) consists of some permutation of \( \pm x, \pm y, \pm z \) where the number of minus signs is 0 or 2.

9. (Putnam 1991, B5). Is there an infinite sequence \( a_0, a_1, a_2, \ldots \) of nonzero real numbers such that for \( n = 1, 2, 3, \ldots \) the polynomial \( p_n(x) = a_0 + a_1x + \cdots + a_nx^n \) has exactly \( n \) distinct real roots?