MA 491 Problem set for October 24, 2006

The Integral

1. Evaluate $\lim_{n\to\infty}\left[\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{3n}\right]$

2. Evaluate $\lim_{n\to\infty}\left[\frac{n}{1^2+n^2}+\frac{n}{2^2+n^2}+\ldots+\frac{n}{n^2+n^2}\right]$

3. Evaluate $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$

4. Evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{k^2+n^2}}$

5. Suppose that f and g are continuous functions on $[0,\, a]$ and that $f(x) = f(a-x)$ and $g(x)+g(a-x) = k$ for all x in $[0,\, a]$, where k is a fixed number. Prove that $\int_o^a f(x)g(x)\,dx = \frac{1}{2}k\int_0^a f(x)\,dx$. Use this fact to evaluate

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx.$$

6. Find all continuous positive functions $f(x)$, for $0 \le x \le 1$ such that $\int_0^1 f(x)\,dx = 1$, $\int_0^1 xf(x)\,dx = a$, $\int_0^1 x^2 f(x)\,dx = a^2$, where a is a given real number.

7. Show that the improper integral

$$\lim_{B\to\infty}\int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

8. Find the maximum value of

$$\int_0^y \sqrt{x^4+(y-y^2)^2}\,dx$$

for $0 \le y \le 1$.

9. Find all real-valued continuously differentiable functions f on the real line such that for all x

$$(f(x))^2 = \int_0^x ((f(t))^2+(f'(t))^2)\,dt + 1990.$$

10. Evaluate $\int_0^a\int_0^b e^{\max\{b^2x^2,a^2y^2\}}\,dy\,dx$, where a and b are positive.