Suggested problems for 10/17/06[1]

1. For $n \geq 1$, let $H(n) = 1 + 1/2 + 1/3 + \ldots + 1/n$. Prove that for $n \geq 2$, $H(n)$ is not an integer.

2. Show that the product of the side lengths of any right triangle with integer lengths is divisible by 60.

3. Prove that every integer is the divisor of infinitely many Fibonacci numbers.

4. Prove that the expressions $2x + 3y$ and $9x + 5y$ are divisible by 17 for exactly the same set of pairs of integers (x, y).

5. Suppose $a_1, a_2, \ldots$ is a sequence of positive integers satisfying $gcd(a_i, a_j) = gcd(i, j)$ for all $i \neq j$. Prove that $a_i = i$ for all i.

6. (Putnam, 1954) Prove that there are no integer solutions to the equation $x^2 + 3xy - 2y^2 = 122$. (See hint below.)

7. Show that it is impossible to tile a square of side length 25 by square tiles of side length 2 and 3.

8. Recall that $\phi(n)$ is the number of positive integers a less than or equal to n such that $gcd(a, n) = 1$. Prove that for any integer n, the sum of $\phi(k)$ over divisors k of n is equal to n.

9. Prove that for all positive integers a, n if $gcd(a, n) = 1$ then $a^{\phi(n)} \equiv 1 (\mathrm{mod} n)$.

10. (Putnam 2000) Prove that for any two integers m, n with $1 \leq m \leq n$, $gcd(m, n)\binom{n}{m}/n$ is an integer.

11. (Putnam 1986) Let Γ consist of all polynomials in x with integer coefficients. For $f, g \in \Gamma$ and m a positive integer, let $f \equiv g (\bmod m)$ mean that every coefficient of $f - g$ is divisible by m. Let n, p be positive integers with p prime. Given that $f, g, h, r, s \in \Gamma$ with $rf + sg \equiv 1 (\mathrm{mod} p)$ and $fg \equiv h (\mathrm{mod} p)$, prove that there exist F and G in Γ with $F \equiv f (\mathrm{mod} p)$, $G \equiv g (\mathrm{mod} p)$ and $FG \equiv h (\mathrm{mod} p^n)$.

12. (Putnam 1998, modified). Let $a_0 = 1$ and for $i \geq 1$ let $a_i = 2^{a_{i-1}}$. Prove that for any integer $n \geq 2$, $a_n - a_{n-1}$ is divisible by n.

13. (Putnam 1991) Let p be an odd prime and let $\mathbb{Z}_p$ denote the field of integers modulo p. How many elements are in the set:

$$\{x^2 : x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : y \in \mathbb{Z}_p\}?$$

[1] Version:10/10/06