MA 491 Problem set #1

1. Determine, with proof, the number of ordered triples $(A_1,\ A_2,\ A_3)$ of sets which have the property that

(1) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
(2) $A_1 \cap A_2 \cap A_3 = \emptyset$

2. Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0\,, 1,\ 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \text{ and } a_m(j+1) = (a_m(j))^2 + 2a_m(j),\ j \geq 0$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

3. Let k be the smallest positive integer with the following property:

There are distinct integers $m_1,\ m_2,\ m_3,\ m_4,\ m_5$ such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients.

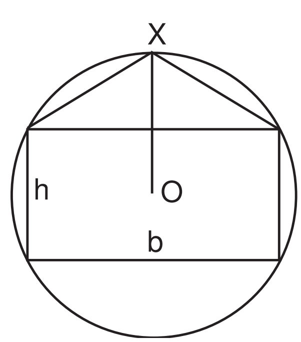
Find, with proof, a set of integers $m_1,\ m_2,\ m_3,\ m_4,\ m_5$ for which the minimum k is achieved.

4. Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the $x-$ and $y-$ axes with diagonal pq. What is the probability that no point of R lies outside of C?

5. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

6. What is the units (i.e. rightmost) digit of $\left[\frac{10^{20000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leq x$.

7. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?



8. Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y,\ y-z,\ z-x)$, where x, y, and z are complex numbers satisfying the simultaneous equations

$$x(x-1)+2yz = y(y-1)+2zx = z(z-1)+2xy$$

and list all such triples T.

9. The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2)=2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

10. For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

11. Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}$$

12. Let R be the region consisting of the points (x, y) in the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.