More review solutions.

1. Let $X$ be a chi-square r.v. with $v$ d.o.f. Its moment generating function is $M_X(t) = (1 - 2t)^{-v/2}$. Hence $M_X'(t) = -v(1 - 2t)^{-v/2 - 1}$ and $M_X''(t) = v(v + 2)(1 - 2t)^{-v/2 - 2}$. Thus \( E[X] = M_X'(0) = \frac{v}{2} \), \( E[X^2] = M_X''(0) = v(v + 2) \) and \( \text{Var}(X) = E[X^2] - (E[X])^2 = \frac{v(v + 2) - v^2}{2} \).

3. By Theorem 8.11, $\bar{X}$ is normal $N(1, \frac{12}{5})$, $\frac{12}{5} S^2$ has the chi-square distribution with 12 d.o.f., and $\bar{X}$ and $S^2$ are independent. Thus

a) \( P\left( \frac{12}{5} \leq \frac{X^2}{0.12} \right) = .99 \), which implies \( P(S^2 \leq 10.92) = .99 \) since \( 10.92 = \frac{5}{12} X^2_{0.12} = 5_{12} (26.217) \). Thus \( \lambda = 10.92 \)

b) We use independence. We want to choose $\lambda$ so that
   \[ 0.95 = P(\bar{X} \leq \lambda, S^2 \leq \theta) = P(\bar{X} \leq \lambda)P(S^2 \leq \theta) = (0.99)P(\bar{X} \leq \lambda). \]

Let $\Phi$ be the c.d.f. of the standard normal. Since $\frac{\bar{X} - 1}{\sqrt{\frac{12}{5} \lambda}}$ is standard normal, we thus require

\[ .95 = \Phi\left( \frac{\lambda - 1}{\sqrt{\frac{5}{12}}} \right) \]

From the normal tables, \( (\lambda - 1)/\sqrt{\frac{5}{12}} \approx 1.75 \) or $\lambda = 1 + \sqrt{\frac{5}{12}} (1.75)$

c) Since $\frac{12}{5} S^2$ is chi-square with 12 d.o.f., the density of $S^2$ is

\[ f_{S^2}(x) = \frac{1}{2\pi} \sqrt{\frac{\lambda}{\pi}} e^{-\frac{\lambda x}{2\pi}} \]

where $f_{S^2}$ is the chi-square density for $\theta = 12$. Using the formula for this density

\[ f_{S^2}(x) = \begin{cases} \frac{12}{5} \frac{1}{2\pi} (\frac{12}{5} x)^{5} e^{-\frac{6}{5} x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \]

d) Using independence and the normal density for $\bar{X}$, the joint density $g(x_1, x_2)$ of $(\bar{X}, S^2)$ is

\[ g(x_1, x_2) = \frac{1}{\sqrt{\frac{12}{5} \pi}} e^{-\frac{(x_1 - 1)^2}{2\pi} - \frac{(x_2 - 12)^2}{12}} \]

It is 0 elsewhere, if $-\infty < x_1 < \infty$, $x_2 > 0$.\]
4. Since $X_{1/2} \sim N(0,1)$ and $Y_{-2} \sim N(0,1)$ If $X$ and $Y$

are independent (I forgot to say this in stating the problem!)

\[(X_{1/2})^2 + (Y_{-2})^2 = \frac{X^2}{4} + \frac{(Y-2)^2}{16}\]

has the chi-square distribution with 2 d.o.f.

Problem on the review sheet.

Since $X_1^2 + \ldots + X_5^2$ is chi-square with 5 d.o.f. and

\[\frac{(X_1^2 + \ldots + X_5^2)/5}{Y_1^2 + \ldots + Y_7^2/7}\]

has the F-dist. with (5,7) d.o.f. Since $f_{0.05,5,7} = 3.97$

\[P\left(\frac{28}{5} R > 3.97\right) = .05\]

and

\[P\left(R > \frac{5}{28} (3.97)\right) = .05\]