FORMULA SHEET

Distributions

A. (Chi-square) A random variable $U$ has the chi-square distribution with $\nu$ degrees of freedom if its density has the form

$$f(x, \nu) = \begin{cases} \frac{1}{2^{\nu/2} \Gamma(\frac{\nu}{2})} x^{\nu/2-1} e^{-x/2}, & \text{if } x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then $E[U] = \nu$, $\text{Var}(U) = 2\nu$, $M(t) = E[e^{tU}] = \frac{1}{(1 - 2t)^{\nu/2}}$.

B. (Normal) $X \sim N(\mu, \sigma^2)$ ($X$ is normal with mean $\mu$ and variance $\sigma^2$) if its distribution is

$$n(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \text{ for } -\infty < x < \infty.$$  

Then $E[X] = \mu$, $\text{Var}(X) = \sigma^2$, $M(t) = e^{\mu t + \sigma^2 t^2/2}$.

- If $X_1, \ldots, X_n$ is a random sample from a Bernoulli distribution with $\theta = P(X_i = 1)$, and $\hat{\theta} = \bar{X}$ is the proportion of 1's (successes), $(\hat{\theta} - \theta)/\sqrt{\theta(1 - \theta)/n}$ is approximately $N(0, 1)$ when $n$ is large.

C. (Exponential) A r.v. $X$ has the exponential distribution with parameter $\theta$ if its density is

$$g(x, \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{for } x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then $E[X] = \theta$, $\text{Var}(X) = \theta^2$, $M(t) = E[e^{tX}] = \frac{1}{1 - \theta t}$.

D. (t-distribution) A r.v. $T_\nu$ has the $t$-distribution with $\nu$ degrees of freedom if its density is

$$f(t, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi \nu} \Gamma(\frac{\nu}{2})} \frac{1}{(1 + t^2/\nu)^{(\nu+1)/2}}.$$  

$T_\nu$ has mean 0 and variance $\nu/(\nu - 2)$.

- If $\bar{X}$ and $S^2$ are the sample mean and variance of a random sample of size $n$ from a $N(\mu, \sigma^2)$ distribution, then $\frac{\bar{X} - \mu}{S/\sqrt{n}}$ has the $t$-distribution with $n - 1$ d.o.f.

E. (F-distribution). A random variable $W$ has the $F$ distribution with $(\nu_1, \nu_2)$ degrees of freedom if its density is

$$f(w; \nu_1, \nu_2) = \frac{\Gamma((\nu_1 + \nu_2)/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2-1} \frac{1}{(1 + \nu_1 w/\nu_2)^{(\nu_1+\nu_2)/2}}.$$  

Its mean is $\nu_2/(\nu_2 - 2)$. For the $F$ distribution $f_{1-\alpha, \nu_1, \nu_2} = \frac{1}{f_{\alpha, \nu_2, \nu_2}}$. 
Examples:

- \( \frac{U/\nu_1}{V/\nu_2} \) has the \( F \)-distribution with \((\nu_1, \nu_2)\) d.o.f., if \( U \) is chi-square with \( \nu_1 \) d.o.f., if \( V \) is chi-square with \( \nu_1 \) d.o.f., and if \( U \) and \( V \) are independent.

- If \( S_1^2 \) is the sample variance of random sample of size \( n_1 \) from \( N(\mu_1, \sigma_1^2) \) and if \( S_2^2 \) is the sample variance of random sample of size \( n_2 \) from \( N(\mu_2, \sigma_2^2) \), then

\[
\frac{\sigma_2 S_1^2}{\sigma_1 S_2^2} \text{ has the } F \text{-distribution with } (n_1 - 1, n_2 - 1) \text{ d.o.f.}
\]

---

**Transformations of random variables.**

**Case 1:** \( Y = u(X) \).

Suppose there is a unique function \( w \) such that, for \((x, y)\) satisfying \( y = u(x) \) and \( f_X(x) > 0 \), we have \( y = u(x) \) if and only if \( x = w(y) \). Then

\[
f_Y(y) = f_X(w(y))|w'(y)|,
\]

This is valid for \( y \) in the range of \( u(x) \) for \( x \) such that \( f_X(x) > 0 \). Otherwise \( f_Y(y) = 0 \).

**Case 2:** \( Y = u(X_1, X_2) \) where \((X_1, X_2)\) has joint density \( f(x_1, x_2) \).

Suppose there is a unique function \( w(x_1, y) \) such that, for \((x_1, x_2, y)\) satisfying \( y = u(x_1, x_2) \) and \( f(x_1, x_2) > 0 \), we have \( y = u(x_1, x_2) \) if and only if \( x_2 = w(x_1, y) \). Then, the joint density of \((X_1, Y)\) is

\[
g(x_1, y) = f(x_1, w(x_1, y))\left| \frac{\partial w}{\partial y}(x_1, y) \right|,
\]

for \((x_1, y)\) such that \( y = u(x_1, x_2) \) where \( f(x_1, x_2) > 0 \). Elsewhere \( g(x_1, y) = 0 \). The density of \( Y \) is

\[
f_Y(y) = \int_{-\infty}^{\infty} f(x_1, w(x_1, y))\left| \frac{\partial w}{\partial y}(x_1, y) \right| dx_1.
\]

**Case 3:** \((Y_1, Y_2) = (u_1(X_1, X_2), u_2(X_1, X_2))\), where \((X_1, X_2)\) has joint density \( f(x_1, x_2) \).

The joint density of \((Y_1, Y_2)\) is

\[
g(y_1, y_2) = f(u_1(y_1, y_2), u_2(y_1, y_2)) |J(y_1, y_2)|,
\]

where

\[
J(y_1, y_2) = \det \begin{pmatrix} \frac{\partial w_1}{\partial y_1} & \frac{\partial w_1}{\partial y_2} \\ \frac{\partial w_2}{\partial y_1} & \frac{\partial w_2}{\partial y_2} \end{pmatrix},
\]

and where \((x_1, x_2) = (w_1(y_1, y_2), w_2(y_1, y_2))\) if and only if \((y_1, y_2) = (u_1(x_1, x_2), u_2(x_1, x_2))\), for \((y_1, y_2)\) in the range of values of \((u_1(x_1, x_2), u_2(x_1, x_2))\) for which \( f(x_1, x_2) > 0 \). The formula for \( g \) is valid on this range and elsewhere it is 0.