12.3. The hypergeometric distribution for parameters $N = 7$, $n = 2$, and $k$ is the probability mass function (see page 176 with $M$ replaced by $k$)

$$h(x; k) = \frac{{k \choose x} {7-k \choose 2-x}}{{7 \choose 2}},$$

for non-negative integers $x$ in $\{0, 1, 2\}$ such that $x \leq k$ and $2 - x \leq 7 - k$.

(This is the probability that a random sample of size 2 without replacement from a box containing 7 marbles, $k$ of which are red and $7 - k$ of which are green, contains $x$ red marbles.)

The aim is to test $H_0 : k = 2$ versus $H_1 : k = 4$, using the test which rejects $H_0$ if $X$ (the number of red balls in the sample) is equal to 2. The type I error is thus the probability of 2 red balls in a sample of size 2 given that the box contains $k = 2$ and this is

$$\alpha = h(2; 2) = \frac{{5 \choose 0}}{{7 \choose 2}} = \frac{1}{21}.$$  

The probability of type II error is the probability of fewer than 2 red balls in a sample of size 2 from a box containing 4 red balls. This is

$$\beta = 1 - h(2; 4) = 1 - \frac{{4 \choose 0} {3 \choose 0}}{{7 \choose 2}} = 1 - \frac{6}{21} = \frac{15}{21} = \frac{5}{7}.$$  

12.4. Refering to Example 12.1,

$$\alpha = P(X \leq 16; \theta = 0.90) = P(X \leq 14; \theta = 0.90) + P(X = 15; \theta = 0.90) + P(X = 16; \theta = 0.90)$$

Since $P(X = 15; \theta = 0.90) = P(X = 5; \theta = 0.10) = 0.0319$ (use Table I for $n = 20$) and similarly $P(X = 16; \theta = 0.90) = P(X = 4; \theta = 0.10) = 0.0898$, we get using the result of Example 12.1 that $\alpha = 0.0114 + 0.0319 + 0.0898 = 0.1331$.

For the type II error,

$$\beta = P(X > 16; \theta = 0.60) = P(X > 14; \theta = 0.60) - P(X = 15; \theta = 0.60) - P(X = 16; \theta = 0.60)$$

$$= 0.1255 - 0.746 - 0.350 = 0.0159.$$  

(In this calculation we see clearly the trade-off between type I and type II error probabilities. Compared to the test proposed in Example 12.1, the type I error probability has been increased substantially, while the type II error probability has been decreased.)

12.5. For a geometric r.v. with parameter $\theta$, $P(X = x; \theta) = \theta(1 - \theta)^{x-1}$, $x = 1, 2, \ldots$, and $P(X \geq x; \theta) = (1 - \theta)^{x-1}$. Consider $H_0 : \theta = \theta_0$, versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$. For a rejection region of the form $X \geq k$, $\alpha = P(X \geq k; \theta_0) = (1 - \theta_0)^{k-1}$ and $\beta = P(X < k; \theta_1) = 1 - (1 - \theta_1)^{k-1}$.
12.6. The density of a geometric r.v. with mean $\theta$ is $f(x; \theta) = (1/\theta)e^{-x/\theta}1_{(0, \infty)}(x)$. Consider testing $H_0: \theta = 2$ versus $H_1: \theta = 5$ from a single observation $X$ with the test that rejects $H_0$ if $X > 3$. Then
\[
\alpha = P(X > 3; \theta = 2) = \int_3^\infty \left(\frac{1}{2}\right)e^{-x/2} \, dx = e^{-3/2}.
\]
While
\[
\beta = P(X \leq 3; \theta = 5) = \int_0^3 \left(\frac{1}{5}\right)e^{-x/5} \, dx = 1 - e^{-3/5}.
\]

12.7. Let $(X_1, X_2)$ be a random sample of size 2 from a normal population with unknown mean $\mu$ and known variance $\sigma^2 = 1$. Let $\mu_1 > \mu_0$. Test $H_0: \mu = \mu_0$ versus $H_1: \mu = \mu_1$ by rejecting $H_0$ if $\bar{X} = (X_1 + X_2)/2 > \mu_0 + 1$. If $\mu_0$ is the mean, then $\bar{X}$ is $N(\mu_0, 1/2)$ and thus $\sqrt{2}(\bar{X} - \mu_0)$ is a standard normal random variable. Thus,
\[
\alpha = P(\bar{X} > \mu_0 + 1; \mu = \mu_0) = P \left(\sqrt{2}(\bar{X} - \mu_0) > \sqrt{2}; \mu = \mu_0\right) = 1 - \Phi(\sqrt{2}) = 0.0787,
\]
where $\Phi$ is the distribution function for the standard unit normal.