Note. If $A$ is a subset of the real numbers, we define the indicator function

$$1_A(x) = \begin{cases} 1, & \text{if } x \text{ is in } A; \\ 0, & \text{if not.} \end{cases}$$

7.1. In this problem $X$ has density $f_X(x) = 2xe^{-x^2}1_{(0,\infty)}(x)$, and $Y = X^2$. Note that $P(Y > 0) = 1$ so we can take $f_Y(y) = 0$ if $y \leq 0$. Since the function $u(x) = x^2$ restricted to the domain $x > 0$ is invertible with inverse $w(y) = \sqrt{y}$ on the domain $\{y > 0\}$, the formula of Theorem 7.1 says

$$f_Y(y) = f_X(\sqrt{y}) \frac{d}{dy}(\sqrt{y}) = 2\sqrt{y}e^{-y} \frac{1}{2\sqrt{y}} = e^{-y}, \quad y > 0.$$ 

In summary, $f_Y(y) = e^{-y}1_{(0,\infty)}(y)$.

7.5. Since $X_1$ and $X_2$ are independent, their joint density is

$$f(x_1,x_2) = \frac{1}{\theta_1 \theta_2} e^{-x_1/\theta_1 - x_2/\theta_2}1_{(0,\infty)}(x_1)1_{(0,\infty)}(x_2)$$

If $Y = X_1 + X_2$, then following Example 7.3, if $y > 0,$

$$F(y) = \int_0^y \left[ \int_0^{y-x_2} \frac{1}{\theta_1 \theta_2} e^{-x_1/\theta_1 - x_2/\theta_2} dx_1 \right] dx_2$$

$$= \int_0^y \frac{1}{\theta_2} e^{-x_2/\theta_2} [1 - e^{-(y-x_2)/\theta_1}] dx_2$$

$$= 1 - \theta_2/\theta_1 e^{-y/\theta_2} - \theta_1/\theta_2 e^{-y/\theta_1}.$$ 

Thus, taking a derivative, for $y > 0$, $f_Y(y) = \frac{1}{\theta_2 - \theta_1} [e^{-y/\theta_2} - e^{-y/\theta_1}]$. For $y \leq 0$, $f_Y(y) = 0$.

Of course this last calculation assumed $\theta_1 \neq \theta_2$. If $\theta_1 = \theta_2 = \theta$, then,

$$F(y) = \int_0^y \left[ \int_0^{y-x_2} \frac{1}{\theta^2} e^{-x_1/\theta - x_2/\theta} dx_1 \right] dx_2$$

$$= \int_0^y \frac{1}{\theta} e^{-x_2/\theta} [1 - e^{-(y-x_2)/\theta}] dx_2$$

$$= 1 - e^{-y/\theta} - \frac{y}{\theta} e^{-y/\theta}.$$
Taking a derivative, for $y$, $f_Y(y) = \frac{y}{\theta^2} e^{-y/\theta}$. This is the density of a gamma distribution with parameters $\alpha = 2$ and $\beta = \theta$. See section 6.3.

7.16. Let the density of $X$ be $f_X(x) = x/2$ for $0 < x < 2$, and otherwise zero; that is $f_X(x) = (x/2)1_{(0,2)}(x)$. Let $Y = X^3$. Since $P(0 < X < 2) = 1$, it follows that $P(0 < Y < 8) = 1$ and thus that we can set $f_Y(y) = 0$ if $y \leq 0$ or $y \geq 8$. The function $y = x^3$ has the inverse $w(y) = y^{1/3}$. Therefore, by Theorem 7.1, for $0 < y < 8$,

$$f_Y(y) = f_X(w(y)) \left| \frac{dy}{dx} \right| = \frac{y^{1/3} y^{-2/3}}{2} = \frac{1}{6y^{1/3}}.$$

In summary, for any $y$, $f_Y(y) = \frac{1}{6y^{1/3}} 1_{(0,8)}(y)$.

7.17. Let the density of $K$ be

$$f_X(x) = \begin{cases} \frac{kx^3}{(1+2x)^6}, & \text{for } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = \frac{2X}{1+2X}$. Notice that the range of the function $u(x) = 2x/(1+2x)$ restricted to the domain $x > 0$ is $0 < y < 1$. The solution to $y = u(x)$ (when $y \neq 1$) is $x = w(y) = y/(2(1-y))$. Observe that $w'(y) = 1/[2(1-y)^2]$. Thus, the density of $Y$ is

$$f_Y(y) = f_X(w(y)) \frac{1}{2(1-y)^2} = \frac{k}{16} y^3 (1-y), \quad \text{if } 0 < y < 1.$$ 

Thus $Y$ must have the beta distribution with $\alpha = 4$ and $\beta = 2$ (see section 6.4), which means

$$\frac{k}{16} = \frac{\Gamma(4+2)}{\Gamma(4)\Gamma(2)} = \frac{5!}{3!\cdot1!} = 20.$$ 

Hence $k = 320$. (We could also easily find this $k$ by setting $\int_0^1 k/16 y^3 (1-y) dy = 1.$)

7.20(a). Let $X$ have the density $f(x) = [3x^2/2]1_{(-1,1)}(x)$. Let $Y = |X|$. Since $P(-1 < X < 1) = \text{and } P(X = 0) = 0$, it follows that $P(0 < Y < 1) = 1$ and hence we can set $f_Y(y) = 0$ if $y \leq 0$ or $y \geq 1$. For $0 < y < 1$, then, using symmetry of $f$,

$$f_Y(y) = \frac{d}{dy} P(|X| \leq y) = \frac{d}{dy} \int_{-y}^{y} 3x^2/2 \, dx$$

$$= \frac{d}{dy} 2 \int_{0}^{y} 3x^2/2 \, dx = 3y^2.$$ 

2
7.30. Using the notation \(1_{(0,1)}(x)\) for the function which equals 1 when \(0 < x < 1\) and 0 elsewhere, the joint density of \((X,Y)\) is given as \(f(x,y) = 12xy(1-y)1_{(0,1)}(x)1_{(0,1)}(y)\). Let \(Z = u(X,Y) = XY^2\). For each \(y\), we can invert \(z = u(x,y)\) to express \(x\) as a function of \(y\) and \(z\): \(x = w(y,z) = z/y^2\). By the method of section 7.3, the joint density of \((y,z)\) is

\[
f(w(y,z),y)|\frac{\partial w}{\partial z}| = \frac{12z(1-y)}{y^3}1_{(0,1)}(z/y^2)1_{(0,1)}(y),
\]

and, since \(1_{(0,1)}(z/y^2) = 0\) if \(z \leq 0\) or \(y \leq \sqrt{z}\),

\[
f_Z(z) = \int_{-\infty}^{\infty} f(w(y,z),y)|\frac{\partial w}{\partial z}| \, dy = 1_{(0,1)}(z) \int_{\sqrt{z}}^{1} \frac{12z(1-y)}{y^3} \, dy = [6z + 6 - 12\sqrt{z}] 1_{(0,1)}(z).
\]

7.34. In this problem the density is \(f(x,y) = (1/2)1_{(0,\infty)}(x)1_{(0,\infty)}(y)1_{(0,2)}(x+y)\). To find the probability density of \(U = Y - X\), note that \(Y = X + U\) and the derivative of \(x+u\) with respect to \(u\) is simply 1. Hence the joint density of \((x,u)\) is \(f(x,x+u)\) and the density of \(U\) is

\[
f_U(u) = \int_{-\infty}^{\infty} f(x,x+u) \, dx = \int_{0}^{\infty} (1/2)1_{(0,\infty)}(x+u)1_{(0,2)}(2x+u) \, dx
\]

\[
= \left\{ \begin{array}{ll}
  \int_{0}^{1-u/2} (1/2) \, dx = (1/2) - (u/4), & \text{if } 0 \leq u < 2; \\
  \int_{1-u/2}^{1} (1/2) \, dx = (1/2) + (u/4), & \text{if } -2 \leq u < 0; \\
  0, & \text{otherwise.}
\end{array} \right.
\]

7.36. The joint density of \((X_1, X_2)\) is \(f(x_1, x_2) = 4x_1x_21_{(0,1)}(x_1)1_{(0,1)}(x_2)\). If we let \(y_1 = x_1^2\) and \(y_2 = x_1x_2\), note that, for \(y_1 \geq 0\), \(x_1 = w_1(y_1, y_2) = \sqrt{y_1}\) and \(x_2 = w_2(y_1, y_2) = y_2/\sqrt{y_1}\). The Jacobian of the function \(w\) is

\[
J = |\frac{\partial w_1}{\partial y_1}\frac{\partial w_2}{\partial y_2} - \frac{\partial w_1}{\partial y_2}\frac{\partial w_2}{\partial y_1}| = \frac{1}{2y_1}.
\]

If \(Y_1 = X_1^2\) and \(Y_2 = X_1X_2, Y_1\) and \(Y_2\) must be positive with probability 1, so the joint density \(g(y_1, y_2)\) equals 0 if either \(y_1 \leq 0\) or \(y_2 \leq 0\). For \(y_1 > 0, y_2 > 0,\)

\[
g(y_1, y_2) = \frac{2y_2}{y_1}1_{(0,1)}(\sqrt{y_1})1_{(0,1)}(y_2/\sqrt{y_1})
\]

\[
= \left\{ \begin{array}{ll}
  \frac{2y_2}{y_1}, & \text{if } 0 < y_1 < 1, \ 0 < y_2 < \sqrt{y_1}; \\
  0, & \text{otherwise.}
\end{array} \right.
\]
A random sample of size 81 is drawn from a population with mean \( \mu = 128 \) and standard deviation \( \sigma = 6.3 \).

(a) The variance of \( \bar{X} \) is \( \sigma^2/n = 6.3^2/81 = .49 \). By Chebyshev’s inequality,

\[
P(\bar{X} < 126.6, \bar{X} > 129.4) = P(\left| \bar{X} - 128 \right| > 1.4) \leq \frac{\text{Var}(\bar{X})}{(1.4)^2} = \frac{.49}{(1.4)^2} = \frac{1}{4}.
\]

(b) By the Central Limit Theorem, \( (\bar{X} - \mu)/(\sigma/\sqrt{n}) = (\bar{X} - 128)/.7 \) is approximately distributed like a standard normal. Thus, if \( \Phi(z) \) denotes the standard normal distribution function

\[
P(\bar{X} < 126.6, \bar{X} > 129.4) = P((\bar{X} - 128)/.7 > 2) = 2[1 - \Phi(2)] = 2(.02275).
\]