13.2. Let $\alpha$ and $\beta$ denote the desired levels for the probabilities of type I and type II errors, respectively. The appropriate test statistic for the one-sided test is

$$ Z = \frac{X - \mu_0}{\sigma/\sqrt{n}} $$

and the rejection region should be of the form \{ $Z \geq k$ \}. To obtain a size $\alpha$ test we must set $k = z_\alpha$, since $Z$ is standard normal under the null hypothesis. The type II error of this test is

$$ b = P_{\mu_1} (Z < z_\alpha) = 1 - P_{\mu_1} (Z \geq z_\alpha), \text{ or } P_{\mu_1} (Z \geq z_\alpha) = 1 - b, $$

where $P_{\mu_1}$ means we compute the probability assuming the alternative hypothesis that $\mu = \mu_1$. To solve the problem, we want to choose $n$ so that $b = \beta$. This is done by noting that

$$ Z = \frac{X - \mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}, $$

hence that

$$ Z \geq z_\alpha \text{ is equivalent to } \frac{X - \mu_1}{\sigma/\sqrt{n}} \geq z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}. $$

and noting that

$$ \frac{X - \mu_1}{\sigma/\sqrt{n}} \text{ is standard normal under } H_1. $$

Therefore,

$$ P_{\mu_1} (Z \geq z_\alpha) = P_{\mu_1} \left( \frac{X - \mu_1}{\sigma/\sqrt{n}} \geq z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} \right) = 1 - \Phi \left( z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} \right). $$

This will equal $1 - \beta$ if

$$ z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} = z_{1-\beta}. $$

Remembering that $z_{1-\beta} = -z_\beta$, and solving for $n$ yields

$$ n = \frac{\sigma^2(z_\alpha + z_\beta)^2}{(\mu_1 - \mu_0)^2}. $$

13.9. Let $X_1, \ldots, X_n$ be a random sample from the Poisson distribution with parameter $\lambda$. Recall that the mean of the Poisson distribution is $\lambda$, so it is sensible to use the sample mean, or equivalently, $X_1 + \cdots + X_n$ as a test statistic. For example, to test $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda > \lambda_0$, one
should use a rejection region of the form \( \{X_1 + \cdots + X_n \geq k\} \). From Example 7.15, \( X_1 + \cdots + X_n \), has the Poisson distribution with parameter \( n\lambda \). Therefore to obtain a level \( \alpha \) test for the one-sided alternative, \( \lambda > \lambda_0 \), we would reject if \( X_1 + \cdots + X_n \geq k_\alpha \) where \( k_\alpha \) is the smallest integer such that
\[
P_{\lambda_0} (X_1 + \cdots + X_n \geq k_\alpha) = \sum_{j=k_\alpha}^{\infty} \frac{(n\lambda_0)^j e^{-n\lambda_0}}{j!} \leq \alpha.
\]

13.38. (i) Set-up: We assume that the first survey is a random sample of size 400 from a \( N(\mu_1, (2.4)^2) \) distribution and the second survey of size 500 from a \( N(\mu_2, (2.5)^2) \) distribution. We want to test \( H_0 : \mu_1 - \mu_2 = -0.5 \) against \( H_1 : \mu_1 - \mu_2 < -0.5 \). at the \( \alpha = 0.05 \) level of significance.

(ii) Test statistic and critical region: Let \( \bar{X}_1 \) and \( \bar{X}_2 \) be the sample means of the samples. Our test will reject \( H_0 \) if
\[
Z = \frac{\bar{X}_1 - \bar{X}_2 - (-0.5)}{\sqrt{(2.4)^2/400 + (2.5)^2/500}} \leq -z_{.05} = -1.645.
\]
This will be a level \( \alpha = 0.05 \) test, because, under the null hypothesis, \( Z \) is a standard normal random variable.

(iii) From the data given, the observed value \( z \) of \( Z \) is \( z = \frac{53.8 - 54.5 + 0.5}{\sqrt{(2.4)^2/400 + (2.5)^2/500}} = -1.219. \)

(iv) Since \( -1.214 > -1.645 \), we accept \( H_0 \).

13.40. (i) Assume the first sample of size 6 is from \( N(\mu_1, \sigma^2) \) and the second sample of size 6 is from \( N(\mu_2, \sigma^2) \). Then we know from Chapter 11, Theorem 11.5, that
\[
T = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p \sqrt{1/6 + 1/6}} \text{ has the } T \text{ distribution with } 6 + 6 - 2 = 10 \text{ d.o.f.} \tag{1}
\]
Here \( S_p \) denotes the sample pooled variance \( S_p = \sqrt{(S_1^2 + S_2^2)/2} \), where \( S_1^2 \) and \( S_2^2 \) are the sample variances of the two samples. We wish to test \( H_0 : \mu_1 = \mu_2 \) against \( H_1 : \mu_1 \neq \mu_2 \).

(ii) Since by Theorem 11.5, \( \bar{X}_1 - \bar{X}_2 \pm t_{0.01,10} S_p \sqrt{1/3} \) is a \( 0.01 \times 100\% \) confidence interval. Hence a level \( \alpha = 0.01 \) test for \( H_0 \) rejects \( H_0 \) if 0 is not in this confidence interval.

(iii) For the data, the confidence interval is
\[
[77.4 - 72.2 - (3.169)\sqrt{\frac{(3.3)^2 + (2.1)^2}{2}} \sqrt{\frac{T}{3}}, 77.4 - 72.2 + (3.169)\sqrt{\frac{(3.3)^2 + (2.1)^2}{2}} \sqrt{\frac{T}{3}}] = [0.14, 10.26].
\]

(iv) The confidence interval does not contain 0. Hence, we reject the null hypothesis.

13.47. Recall that for a sample of size \( n \) from a normal \( N(\mu, \sigma^2) \) population, \( (n - 1)S^2/\sigma^2 \) has the chi-square distribution with \( n - 1 \) degrees of freedom. Thus, for this problem, under the null
hypothesis that \( \sigma^2 = 0.01 \), \( 8s^2/(.01)^2 = 8s^2 \times 10^4 \) should have the chi-square distribution with 8 degrees of freedom. Since the alternative hypothesis is that \( \sigma^2 < 0.01 \), an appropriate test at significance level \( \alpha = 0.01 \) is to reject if \( 8s^2 \times 10^4 < \chi_{.99,8}^2 = 1.646 \). In this case, \( s^2 = (.0086)^2 \), and \( 80000 \times (.0086)^2 = 5.92 \). Therefore, we accept \( H_0 \).

13.48. This problem features a two-sided alternative to \( H_0 : \sigma^2 = (250)^2 \). Under the null hypothesis \( 23S^2/(250)^2 \) is a chi-square random variable with 23 degrees of freedom. So we should reject if the value of this statistic is less than \( \chi_{.995,23}^2 = 44.181 \). In fact, the observed sample variance is \( (238)^2 \) and \( 23 \times (238)^2/(250)^2 = 20.84 \). Hence, we accept \( H_0 \).

13.54. Following Example 13.7, the null hypothesis should be rejected if \( s^2_1/s^2_2 \geq f_{.05,5,5} = 5.05 \) or \( s^2_2/s^2_1 \geq f_{.05,5,5} = 5.05 \). In fact, \( s^2_2/s^2_1 = (3.3/2.1)^2 = 2.47 \) and hence we accept \( H_0 \).

13.67. Following Example 13.9, with the continuity correction, we should reject the null hypothesis if
\[
2 - 0.5 - 200(0.9) \leq -z_{.05} = -1.645.
\]
In this case, since \( x = 175 \), \( z = -1.06 \). We accept \( H_0 \).

13.73. Let \( X_A, X_B, X_C, X_D \) be the number of tires that fail in a test of 200 tires of each brand respectively. Our model is that these are independent random variables with \( n = 200 \) and respective probabilities of failure, \( \theta_A, \theta_B, \theta_C, \theta_D \). The null hypothesis of no difference in tire quality translates to \( H_0 : \theta_A = \theta_B = \theta_C = \theta_D \) and the alternative is that they are not all equal.

<table>
<thead>
<tr>
<th></th>
<th>Fail</th>
<th>Not fail</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_A )</td>
<td>26</td>
<td>174</td>
</tr>
<tr>
<td>( x_B )</td>
<td>23</td>
<td>177</td>
</tr>
<tr>
<td>( x_C )</td>
<td>15</td>
<td>185</td>
</tr>
<tr>
<td>( x_D )</td>
<td>32</td>
<td>168</td>
</tr>
</tbody>
</table>

Under the null hypotheses, the MLE of \( \theta = \theta_A = \theta_B = \theta_C = \theta_D \) is \( \hat{\theta} = (26 + 23 + 15 + 32)/800 = 3/25 = 0.12 \). Using this estimate, the expected number of failures for each brand is \( 200(0.12) = 24 \) and the expected number of non failures is 176. The test statistic is
\[
\chi^2 = \sum_{i=1}^{4} \sum_{j=1}^{2} \frac{(f_{ij} - e_{ij})^2}{e_{ij}},
\]
where \( f_{ij} \) is the observed number in each cell and \( e_{ij} \) is the expected number. Our test is to reject if \( \chi^2 \geq \chi^2_{.05,3} = 7.815 \). An equivalent formula for computing \( \chi^2 \), slightly simpler to handle, is
\[ \sum_{i \in \{A,B,C,D\}} \frac{(x_i - 20\hat{\theta})^2}{200\hat{\theta}(1 - \hat{\theta})} \]. For the given data, this is

\[ \chi^2 = \frac{(26 - 24)^2}{21.12} + \frac{(23 - 24)^2}{21.12} + \frac{(15 - 24)^2}{21.12} + \frac{(32 - 24)^2}{21.12} = 7.102. \]

This is smaller than 7.815 and so we accept \( H_0 \) at the \( \alpha = 0.05 \) significance level.

13.77. This is an odd problem, because there is a cell count of 0 in the data. When this occurs, the chi-square approximation to the distribution of the test statistic defined in section 3.7 cannot be trusted—see the discussion at the top of page 420. When this happens it may be appropriate to combine rows or columns before doing the chi-square test, in ways not explained in the text, whose solution is a straightforward application of the ideas and approximations presented in section 3.7. At the least, one must be cautious about interpreting the results.

We shall go ahead and apply the techniques of section 3.7, ignoring the problem of the cell having an entry of 0. We list the data matrix, together with a computations of \( f_i \) (the row sums) and \( f_j \) (the column sums).

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>Avg.</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>7</td>
<td>12</td>
<td>31</td>
</tr>
<tr>
<td>Avg.</td>
<td>35</td>
<td>59</td>
<td>18</td>
</tr>
<tr>
<td>High</td>
<td>15</td>
<td>13</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ f_1 = 57 \quad f_2 = 84 \quad f_3 = 49 \quad n = 190 \]

The test statistic is \( \chi^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{(f_{ij} - e_{ij})^2}{e_{ij}} \), where \( f_{ij} \) is the observed value in cell \((i, j)\) and \( e_{ij} = f_i f_j / n \). (Here we use \( n \) in place of the test’s \( f \).) We reject \( H_0 \) if \( \chi^2 \geq \chi^2_{0.01,3} = 13.277 \).

For convenience, here is a table of the \( e_{ij} \) values:

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>Avg.</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>15</td>
<td>22.11</td>
<td>12.89</td>
</tr>
<tr>
<td>Avg.</td>
<td>33.6</td>
<td>49.52</td>
<td>28.88</td>
</tr>
<tr>
<td>High</td>
<td>8.40</td>
<td>12.38</td>
<td>7.22</td>
</tr>
</tbody>
</table>

Plugging in to the formula for \( \chi^2 \) we obtain, \( \chi^2 = 52.7 \). Even though the chi-square approximation may not be accurate due to the cell with 0 observations, this value of the statistic is so much larger than the level 13.277 that we can feel confident in rejecting \( H_0 \). (A visual inspection of the tables shows how poorly the \( e_{ij} \)'s match the \( f_{ij} \)'s.
This problem is solved exactly like the previous problem. However, its formulation is somewhat different. Here, we will state the null and alternative hypotheses carefully to make the difference clear, but we will omit the detailed calculation of the test statistic.

Let \((X_1^A, X_2^A, X_3^A)\) denote, respectively, the number of items rejected, imperfect but acceptable, and perfect in a randomly selected shipment from vendor \(A\). Let \((X_1^B, X_2^B, X_3^B)\), and \((X_1^C, X_2^C, X_3^C)\) be defined similarly for vendors \(B\) and \(C\). We assume that \((X_1^A, X_2^A, X_3^A)\) has the multinomial distribution with parameters \(n_1 = 12 + 23 + 89 = 124\) and \((\theta_1^A, \theta_2^A, \theta_3^A)\), (see section 5.8; here \(0 \leq \theta_i^A \leq 1\) for each \(i\), \(\theta_1^A + \theta_2^A + \theta_3^A = 1\), and \(\theta_i^A\) represents the probability that a randomly selected item from vendor \(A\) falls in category \(i\).) Similarly, \((X_1^B, X_2^B, X_3^B)\) has the multinomial distribution with parameters \(n_2 = 82\) and \((\theta_1^B, \theta_2^B, \theta_3^B)\), and \((X_1^A, X_2^A, X_3^A)\) has the multinomial distribution with parameters \(n_3 = 170\) and \((\theta_1^C, \theta_2^C, \theta_3^C)\).

The null hypothesis that the vendors ship products of equal quality is \(H_0 : \theta_1^A, \theta_2^A, \theta_3^A = \theta_1^B, \theta_2^B, \theta_3^B = \theta_1^C, \theta_2^C, \theta_3^C\), the alternative hypothesis is that these vectors of probabilities are not all the same.

Despite the difference in the set-up, the chi-square test as for the \(r \times c\) table in the preceding problem. The rejection region for this test, since it is prescribed to have level \(\alpha = 0.01\), \(\chi^2 \geq \chi^2_{0.01, 4} = 13.277\). My calculation gives a value of \(\chi^2 = 1.30\). Thus we should accept \(H_0\). To understand why \(\chi^2\) is so small, we list the expected counts \(e_{ij}\): \(e_{11} = 13.5\), \(e_{12} = 21.4\), \(e_{13} = 89.0\), \(e_{21} = 8.9\), \(e_{22} = 14.2\), \(e_{23} = 58.9\), \(e_{31} = 18.5\), \(e_{32} = 29.4\), \(e_{33} = 122.1\). Notice how close these numbers are to the observed values.