Solutions to additional problems

1. The power function is

$$P_\theta (X > 4) = \int_4^\infty \frac{e^{-x/\theta}}{\theta} \, dx = e^{-4/\theta}.$$      

This function is increasing in $\theta$ and satisfies: $\lim_{\theta \downarrow 0} e^{-4/\theta} = 0$, and $\lim_{\theta \rightarrow \infty} e^{-4/\theta} = 1$.

2. The density of a random variable with the exponential ($\theta$) distribution is $f(x) = (1/\theta) e^{-x/\theta}$ for $x > 0$; otherwise it is zero. Its cumulative distribution function for $x > 0$ is

$$F(x) = P(X \leq x) = 1 - e^{-x/\theta}.$$      

If a random variable has the exponential distribution it is positive with probability one. Thus, $0 < Y_2 < Y_7$ with probability one. Hence, the joint density $g(y_2, y_7) > 0$ only if $0 < y_2 < y_7$. Then

$$g(y_2, y_7) \, dy_2 \, dy_7 \approx P\left( Y_2 \in [y_2, y_2 + dy_2], Y_7 \in [Y_7, y_7 + dy_7] \right)$$

Up to first order in $dY_2 dy_7$ this is the same as the probability that of the 10 values $X_1, \ldots, X_{10}$, one is less than $y_2$, one is between $y_2$ and $y_2 + dy_2$, four are between $y_2$ and $y_7$, one is between $y_7$ and $y_7 + dy_7$, and the remaining three are all greater than $y_7$. There are

$$10 \cdot 9 \cdot \binom{6}{4} \cdot 4 = \frac{10 \cdot 9}{4!} \cdot 4 = \frac{10}{4!} 3!$$

arrangements of $X_1, \ldots, X_{10}$ like this; one has first 10 choices for the $X_i$ which lands below $y_2$, then 9 choices for that one ($Y_2$) in $[y_2, y_2 + dy_2]$, then 4 out of 8 choices for the $X_i$ falling between $y_2$ and $y_7$, and finally, 4 remaining choices for the $X_i$ falling in $[y_7, y_7 + dy_7]$. For each such arrangement, the probability that they fall as required is $F(y_2) f(y_2) [F(y_7) - F(y_2)]^4 f(y_7) [1 - F(y_7)]^3 \, dy_2 dy_7$. Therefore, for $0 < y_2 < y_7$,

$$g(y_2, y_7) = \frac{10}{4!3!} F(y_2) f(y_2) [F(y_7) - F(y_2)]^4 f(y_7) [1 - F(y_7)]^3$$

$$= \frac{10}{4!3!} \frac{1}{\theta^2} [1 - e^{-y_2/\theta}] e^{-y_2/\theta} [e^{-y_2/\theta} - e^{-y_7/\theta}]^4 e^{-y_7/\theta} [e^{-y_7/\theta}]^3$$

Otherwise, the density is zero.
3. a) The maximum of $\pi_A(\theta)$ over $\theta$ corresponding to the null hypothesis, $-1 \leq \theta \leq 1$ is 0.05 (there was a typo in the graph and .5 should be 0.05). Hence the size of $Z$ is 0.05. The maximum of $B$ over the null hypothesis is 0.1, so 0.1 is the size of $B$.

b) Test $B$ is uniformly more powerful than test $A$ because $\pi_B(\theta) > \pi_A(\theta)$ for all $\theta$ corresponding to the alternative hypothesis, that is, for all $\theta$ such that $|\theta| > 1$.

c) If one is testing at level 0.1 then $B$ is the test to use because it is uniformly more powerful at this level.

12.10. In this problem, a random sample of size $n$ from a $N(\mu, 1)$ population is to be used to test $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$, where $\mu_1 < \mu_0$. The Neyman-Pearson Lemma says that a rejection region of the form

$$C = \{ \ln \left( \frac{L(\mu_0; x_1, \ldots, x_n)}{L(\mu_1; x_1, \ldots, x_n)} \right) \leq k \}$$

where $k$ is chosen so that the size of the test is $\alpha$ is most powerful of size $\alpha$ for testing $H_0$ against $H_1$. Since $L(\mu; x_1, \ldots, x_n) = (2\pi)^{n/2} \exp\{-1/2 \sum_1^n (x_i - \mu)^2\}$, $C$ has the form,

$$C = \left\{ -\frac{1}{2} \sum_1^n (x_i - \mu_0)^2 + \frac{1}{2} \sum_1^n (x_i - \mu_1)^2 \leq k \right\}$$

However $-\frac{1}{2} \sum_1^n (x_i - \mu_0)^2 + \frac{1}{2} \sum_1^n (x_i - \mu_1)^2 = n\bar{x}(\mu_0 - \mu_1) + (n/2)(\mu_1^2 - \mu_0^2)$. So an equivalent formulation of $C$ is, after a bit of algebra, using the fact that $\mu_0 - \mu_1 > 0$:

$$C = \left\{ \bar{x} \leq \left[ k - (n/2)(\mu_1^2 - \mu_0^2) \right]/(n(\mu_0 - \mu_1)) \right\}.$$ 

However the right hand side is just a constant $k'$. So the final form of the critical region given by the Neyman-Pearson lemma is $\{ \bar{x} \leq k' \}$ for some constant $k'$.

To get a test of size $\alpha$ we must choose $k'$ so that $P_{\mu_0}(\bar{X} \leq k') = \alpha$. If $\mu_0$ is the true mean, $\sqrt{n}(\bar{X} - \mu_0)$ is a standard normal random variable. Thus we require that

$$\alpha = P_{\mu_0}(\bar{X} \leq k') = P_{\mu_0}(\sqrt{n}(\bar{X} - \mu_0) \leq \sqrt{n}(k' - \mu_0)) = \Phi(\sqrt{n}(k' - \mu_0)),$$

where $\Phi$ is the cumulative distribution function of the standard normal. This requires that $\sqrt{n}(k' - \mu_0) = -z_\alpha$, or $k' = \mu_0 - (1/\sqrt{n})z_\alpha$. To summarize, the most powerful test of size $\alpha$ consists in using the rejection region $\bar{x} \leq \mu_0(1/\sqrt{n})z_\alpha$.

12.11. The likelihood function of a random sample of size $n$ from an exponential population with mean $\theta$ is $L(\theta; x_1, \ldots, x_n) = (1/\theta^n) \exp\{-1/\theta \sum_1^n x_i\} = (1/\theta^n)e^{-nx/\theta}$, on the set where
The most powerful critical region of size $\alpha$ to test $\theta = \theta_0$ against $\theta = \theta_1$ has the form

$$\left\{ \ln \frac{L(\theta_0; x_1, \ldots, x_n)}{L(\theta_1; x_1, \ldots, x_n)} = n \ln \left( \frac{1}{\theta_0} - \frac{1}{\theta_1} \right) \leq k \right\}.$$

If $\theta_0 < \theta_1$, $(1/\theta_0) - (1/\theta_1) > 0$, and so this critical region has the form

$$\{ \bar{x} \geq k' \}.$$

for some constant $k'$.

To find a test of size $\alpha$ we must choose $k'$ so that $\alpha = P_{\theta_0} (\bar{X} \geq k')$. Now, if $X_1, \ldots, X_n$ are independent, exponential random variables with parameter $\theta_0$ it follows from Example 7.16 that $X_1 + \cdots + X_n$ has a gamma distribution with $\alpha = n$ and $\beta = \theta_0$. Using the density of the gamma distribution with $\alpha = n$, $\beta = \theta_0$, $P_{\theta_0} (\bar{X} \geq k') = P_{\theta_0} (X_1 + \cdots + X_n \geq nk') = \int_{nk'}^{\infty} y^{n-1}e^{-y/\theta_0} \theta_0^n (n-1)! \, dy.

Using a change of variable $x = y/\theta_0$ in the integral, to get a test of size $\alpha$ requires choosing $k'$ so that

$$\alpha = \frac{1}{(n-1)!} \int_{nk'}^{\infty} x^{n-1}e^{-x} \, dx.$$

12.12. The likelihood function of a binomial random variable with parameters $n$ and $\theta$, $0 < \theta < 1$, is $L(\theta; x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}$. We are interested in testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 < \theta_0$. The Neyman-Pearson lemma implies that a test whose rejection region is of the form

$$\left\{ \ln \frac{L(\theta_0; x)}{L(\theta_1; x)} = x \ln \left[ \frac{\theta_0(1 - \theta_1)}{\theta_1(1 - \theta_0)} \right] + n \ln \left[ \frac{1 - \theta_0}{1 - \theta_1} \right] \leq k \right\}$$

will be most powerful at its size. Since $\theta_1 < \theta_0$, $\frac{\theta_0(1 - \theta_1)}{\theta_1(1 - \theta_0)} > 1$ and so $\ln \left[ \frac{\theta_0(1 - \theta_1)}{\theta_1(1 - \theta_0)} \right] > 0$. This implies that the Neyman-Pearson rejection region is of the form

$$\{ x \leq k' \}$$

for some level $k'$. To construct a most powerful test of size $\alpha$, one would choose $k'$ so that $P_{\theta_0} (X \leq k') = \alpha$. Because $X$ is discrete, this will not be possible for all $\alpha$, $0 < \alpha < 1$. In this case one might instead choose the largest $k'$ so that $P_{\theta_0} (X \leq k') \leq \alpha$. When $n$ is large, as in problem 12.13 next, we can approximate the binomial by a normal and choose a test whose type I error probability is very close to $\alpha$. 

3
12.13. In 12.12, let \( n = 100 \) and \( \theta_0 = 0.40 \), and \( \theta_1 = 0.30 \). If \( \theta_0 \) is the value of \( \theta \), then \( \frac{(X - 40)}{\sqrt{100(0.4)(0.6)}} \) is approximately a standard unit normal. Then to get a size \( \alpha = 0.05 \) test we need (using the normal approximation to the binomial with a continuity correction)

\[
0.05 = P_{\theta_0=0.40}(X \leq k') = P_{\theta_0=0.40}\left(\frac{X - 40}{2\sqrt{6}} \leq \frac{k' - 40}{2\sqrt{6}}\right) \approx \Phi\left(\frac{k' + 0.5 - 40}{2\sqrt{6}}\right).
\]

This is achieved by setting

\[
\frac{k' - 40}{2\sqrt{6}} = -z_{0.05} = -1.645 \quad \text{or} \quad k' = 39.5 - (1.645) \cdot 2\sqrt{6} = 31.44.
\]

However, \( X \) is integer valued, so to insure that the type I error probability does not exceed \( 0.05 \), we let \( k' = 31 \).

The type II error is \( P_{\theta_1=0.3}(X \geq 32) \). When \( \theta_1 = 0.3 \), \( \frac{(X - 30)}{\sqrt{100(0.3)(0.7)}} \) is approximately standard normal. Thus, again using the normal approximation with continuity correction

\[
\beta = P_{\theta_1=0.3}(X \geq 32) = P_{\theta_1=0.3}\left(\frac{X - 30}{\sqrt{21}} \geq \frac{1.5}{\sqrt{21}}\right) \approx 1 - \Phi(0.327) = 0.37.
\]

12.15. For a random sample of size \( n \) from a normal \( N(0, \sigma^2) \) population, the likelihood function is

\[
L(\sigma; x_1, \ldots, x_n) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \cdot \frac{1}{\sigma^n} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2\right\}.
\]

Thus,

\[
\ln \lambda = \ln \frac{L(\sigma_0; x_1, \ldots, x_n)}{L(\sigma_1; x_1, \ldots, x_n)} = n[\ln \sigma_1 - \ln \sigma_0] + \left(\sum_{i=1}^{n} x_i^2\right) \left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right).
\]

If \( \sigma_1 > \sigma_0 \), then the coefficient multiplying \( \sum_{i=1}^{n} x_i^2 \) in this formula is negative and so the inequality \( \{\lambda \leq k\} \) is equivalent to an inequality of the form \( \sum_{i=1}^{n} x_i^2 \geq k' \) for some different constant \( k' \). Thus, the most powerful test of size \( \alpha \) for the null hypothesis \( \sigma = \sigma_0 \) against \( \sigma = \sigma_1 \), where \( \sigma_1 > \sigma_0 \) will be defined by the rejection region

\[
\{\sum_{i=1}^{n} x_i^2 \geq k'\},
\]

where \( k' \) is chosen so that

\[
P_{\sigma_0}\left(\sum_{i=1}^{n} X_i^2 \geq k'\right) = \alpha.
\]

If \( \sigma_0 \) is the true parameter, \( X_i/\sigma_0 \) will be normal with mean zero and variance 1 and hence

\[
\frac{1}{\sigma_0^2} \sum_{i=1}^{n} X_i^2 \quad \text{will be a chi-square random variable with} \ n \ \text{degrees of freedom}.
\]
The derivative of $H$ are testing 12.20.

(a) Let $\theta$. We know that the maximum likelihood estimate of $\theta$ is the $\pi$ distribution.

The power function is $\text{power function is } \pi(\theta) = P_\theta(X \leq 15)$, where for each $\theta$, $X$ should be interpreted as a binomial random variable with $n = 20$ and the probability of success being equal to $\theta$. Thus,

\[
\pi(\theta) = P_\theta(X \leq 14) + P_\theta(X = 15) = P_\theta(X \leq 14) + \binom{20}{15} \theta^{15}(1 - \theta)^5.
\]

But $P_\theta(X \leq 14)$ is exactly what is computed in Example 12.5. So using those results: $\pi(.95) = 0.0003 + \binom{20}{15}(.95)^{15}(.05)^5 = 0.0003 + 0.0022 = 0.0025; \pi(.9) = 0.0114 + 0.0319 = 0.0433; \pi(.85) = 0.0674 + 0.1028 = 0.1702; \pi(.8) = 0.1958 + 0.1746 = 0.3704; \text{etc.}$

12.18. The power function is $\pi(\theta) = P_\theta(X \leq 15)$, where for each $\theta$, $X$ should be interpreted as a binomial random variable with $n = 20$ and the probability of success being equal to $\theta$. Thus,

\[
\pi(\theta) = P_\theta(X \leq 14) + P_\theta(X = 15) = P_\theta(X \leq 14) + \binom{20}{15} \theta^{15}(1 - \theta)^5.
\]

But $P_\theta(X \leq 14)$ is exactly what is computed in Example 12.5. So using those results: $\pi(.95) = 0.0003 + \binom{20}{15}(.95)^{15}(.05)^5 = 0.0003 + 0.0022 = 0.0025; \pi(.9) = 0.0114 + 0.0319 = 0.0433; \pi(.85) = 0.0674 + 0.1028 = 0.1702; \pi(.8) = 0.1958 + 0.1746 = 0.3704; \text{etc.}$

12.20. (a) Let $X$ denote the observed number of successes in $n$ trials. We have that

\[
L(\theta; x) = P(X = x; \theta) \binom{n}{x} \theta^x (1 - \theta)^{n-x}, \quad 0 \leq x \leq n.
\]

We know that the maximum likelihood estimate of $\theta$ is the $X/n$ of the number of successes. We are testing $H_0 : \theta = 1/2$ against $H_1 : \theta \neq 1/2$. Thus the likelihood ratio statistic is

\[
\lambda(x) = \frac{L(1/2; x)}{L(x/n; x)} = \frac{(1/2)^n}{(x/n)^x (1 - (x/n))^{n-x}}.
\]

(b) The log-likelihood ratio statistic is $\ln \lambda(x) = n \ln(1/2) + n \ln n - x \ln x - (n-x) \ln(n-x)$. A likelihood ratio test is defined by a critical region of the form $n \ln(1/2) + n \ln n - x \ln x - (n-x) \ln(1-\theta) \leq k$, or, rearranging, $x \ln x + (n-x) \ln(1-\theta) \geq -k + n \ln(1/2) + n \ln n$. The right-hand side is just a constant $K$ independent of $x$, so likelihood ratio tests have critical regions of the form

\[
x \ln x + (n-x) \ln(n-x) \geq K.
\]

(c) Let $f(x) = x \ln x + (n-x) \ln(n-x)$ for $0 < x < n$. This function is symmetric about $x = n/2$; that is

\[
f(n/2 + y) = (n/2 + y) \ln(n/2 + y) + (n/2 - y) \ln(n/2 - y) = f(n/2 - y).
\]

The derivative of $f$ is

\[
f'(x) = \ln x - \ln(n-x) = \ln \left[ \frac{x}{n-x} \right].
\]
For $0 < x < n/2$, $x/(n-x) < 1$ and hence $f'(x) < 0$, while for $n/2 < x < 1$, $f'(x) < 0$. Therefore the graph of $y = f(x)$ is decreasing on $(0, n/2)$ and increasing on $(n/2, \infty)$. If we add to this the symmetry of $f$ about $x = n/2$ we see that a region of the form $\{x; f(x) < K\}$ will have the form of an interval that is symmetric about $x = n/2$; that is, $\{x; f(x) < K\} = \{x; \vert x - n/2 \vert < K'\}$ for some $K'$. Therefore the rejection region $\{x; f(x) \geq K\} = \{x; \vert x - n/2 \vert \geq K'\}$ (Note that the $K$ in the statement of part (c) is different than the constant $K$ in part (b)–we have used $K$ and $K'$ to distinguish them.)

12.21. (a) For this problem we use the fact that the maximum likelihood estimate of the exponential population with unknown mean $\theta$ is the sample mean $\bar{X}$. The likelihood function for a random sample of size $n$ is $L(\theta; x_1, \ldots, x_n) = \theta^{-n} e^{-\bar{x}/\theta}$, where $\bar{x} = (1/n) (x_1 + \cdots + x_n)$. For testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_1$, the likelihood ratio statistic is

$$\lambda(x_1, \ldots, x_n) = \frac{\theta_0^{-n} e^{-\bar{x}/\theta_0}}{(\bar{x})^{-n} e^{-n}}.$$  

(b) Some algebraic manipulation shows that

$$\lambda(x_1, \ldots, x_n) = \left[ \frac{\bar{x} e^{-\bar{x}/\theta_0}}{e \theta_0} \right]^n.$$  

Thus, the inequality $\lambda(x_1, \ldots, x_n) \leq k$ is equivalent to $\bar{x} e^{-\bar{x}/\theta_0} \leq k^{1/n} e \theta_0$, which is of the form

$$\bar{x} e^{-\bar{x}/\theta_0} \leq K,$$

for some constant $K$. This will be the form of the rejection region of the likelihood ratio test.

12.22. (This problem mistakenly refers to Example 10.17, when Example 10.18 is meant.) In this problem, $L(\mu, \sigma^2; x_1, \ldots, x_n) = (2\pi \sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}$. Both the null and alternative hypotheses are composite in this problem. The null hypothesis is that $\mu = \mu_0$ and $\sigma^2 > 0$ is unknown, while the alternative is that $\mu \neq \mu_0$ and $\sigma^2 > 0$ is unknown.

The maximum likelihood estimates of $\mu$ and $\sigma^2$ under the null hypothesis are $\hat{\mu} = \mu_0$ (since $\mu = \mu_0$ is given in the null hypothesis) and the maximum likelihood estimate of the variance for a given mean was found in problem 10.62 to be $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2$.

On the other hand, the maximum likelihood estimates for $\mu$ and $\sigma^2$ both unknown are $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{X})^2$. Therefore,

$$\lambda(x_1, \ldots, x_n) = \frac{L(\mu_0, \hat{\sigma}^2; x_1, \ldots, x_n)}{L(\hat{\mu}, \hat{\sigma}^2; x_1, \ldots, x_n)}$$
\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x} + \bar{x} - \mu_0)^2 = \hat{\sigma}^2 + (\bar{x} - \mu_0)^2 \]

Also \( \hat{\sigma}^2 = [(n - 1)/n]s^2 = (n - 1)[s/\sqrt{n}]^2 \), where \( s^2 \) is the sample variance. Therefore, substituting these identities into the expression obtained for \( \lambda \)

\[ \lambda(x_1, \ldots, x_n) = \left[ \frac{\hat{\sigma}^2 + (\bar{x} - \mu_0)^2}{\hat{\sigma}^2} \right]^{-n/2} = \left[ 1 + \frac{(\bar{x} - \mu_0)^2}{(n-1)(s/\sqrt{n})^2} \right]^{-n/2} = \left[ 1 + \frac{t^2}{n-1} \right]^{-n/2} \]

where \( t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \).

12.24. In this problem, the likelihood function is the same as in 12.22. Also, the maximum likelihood estimates \( \hat{\mu} \) and \( \hat{\sigma}^2 \), are the same as in 12.22 because they are the MLE’s with \( \mu \) and \( \sigma^2 \) unknown. In this problem the null hypothesis is that \( \sigma^2 = \sigma_0^2 \). Hence \( \hat{\sigma}^2 = \sigma_0^2 \). Since \( \mu \) is unknown under the null hypothesis, \( \hat{\mu} = \bar{X} \). Therefore the likelihood ratio statistic is

\[ \lambda(x_1, \ldots, x_n) = \frac{L(\bar{X}, \sigma_0^2; x_1, \ldots, x_n)}{L(\bar{X}, \hat{\sigma}^2; x_1, \ldots, x_n)} = \left[ \frac{\sigma_0}{\hat{\sigma}} \right]^{-n/2} \exp\left\{ \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i - \bar{X})^2 - \frac{1}{2\sigma_0^2} \sum_{i=1}^{n} (x_i - \bar{X})^2 \right\} = \left[ \frac{\sigma_0^2}{\hat{\sigma}^2} \right]^{-n/2} \exp\left\{ \frac{n}{2} - \frac{1}{2\sigma_0^2} n\hat{\sigma}^2 \right\} = \left[ \frac{n\sigma_0^2}{(n-1)s^2} \right]^{-n/2} \exp\left\{ \frac{n}{2} - \frac{1}{2\sigma_0^2} (n-1)s^2 \right\}, \]

where \( s^2 \) is the sample variance. Notice that this statistic is a function of the ratio of \( \sigma_0^2 \) to the sample variance \( s^2 \):

\[ \ln \lambda(x_1, \ldots, x_n) = \frac{n}{2} \ln\left\{ \frac{n-1}{n} \right\} + \frac{n}{2} \ln\left\{ \frac{s^2}{\sigma_0^2} \right\} - \frac{1}{2} \frac{s^2}{\sigma_0^2} + \frac{n}{2}. \]

12.25. Independent random samples of size \( n_1, \ldots, n_k \) from \( k \) normal populations with unknown means and variances are to be used to test the null hypothesis that \( \sigma_1 = \cdots = \sigma_k \) against the alternative that the variances are not all equal.
Let \( x_{1,1}, \ldots, x_{1,n_1} \) by the observed values from population 1, \( x_{2,1}, \ldots, x_{2,n_2} \) the observed values from population 2, etc. Let \( z \) be the vector of all observed values from all populations. Since the populations are sample randomly the likelihood function of all the parameters is

\[
L(\mu_1, \ldots, \mu_k, \sigma_1^2, \ldots, \sigma_k^2; z) = \prod_{i=1}^{k} L(\mu_i; \sigma_i; x_{i,1}, \ldots, x_{i,n_i}).
\]

where \( L(\mu, \sigma^2; y_1, \ldots, y_m) \) is the normal likelihood as used in the previous problems.

(a) We want to compute the maximum likelihood estimates of the means and variance under the null hypothesis of equal variances. When the null hypothesis holds with common variance \( \sigma^2 \) the likelihood function is

\[
L(\mu_1, \ldots, \mu_k; \sigma^2; z) = \prod_{i=1}^{k} L(\mu_i; \sigma_i; x_{i,1}, \ldots, x_{i,n_i})
\]

Whatever the value of \( \sigma^2 \), the term \( L(\mu_i; \sigma^2; x_{i,1}, \ldots, x_{i,n_i}) \) is maximized over \( \mu_i \) by setting \( \hat{\mu}_i = \bar{x}_i = (1/n_i)\sum x_{i,j} \). Thus these choices will maximize the likelihood of the product. To find the maximum likelihood estimate of \( \sigma^2 \), we substitute the \( \hat{\mu}_i \) in (1) and maximize with respect to \( \sigma^2 \). To do this, note that

\[
L(\hat{\mu}_i, \sigma^2; x_{i,1}, \ldots, x_{i,n_i}) = (2\pi\sigma^2)^{-n_i/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{n_i} (x_{i,j} - \bar{x}_i)^2 \right\} = (2\pi\sigma^2)^{-n_i/2} \exp\left\{ -\frac{(n_i - 1)s_i^2}{2\sigma^2} \right\},
\]

where \( s_i^2 \) is the sample variation of population \( i \). Thus, setting \( n = \sum_{i=1}^{k} n_i \),

\[
L(\hat{\mu}_1, \ldots, \hat{\mu}_k, \sigma^2; z) = (2\pi\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{k} (n_i - 1)s_i^2 \right\}.
\]

The log of this is \(-n/2 \ln(2\pi) - (n/2) \ln(\sigma^2) - (1/2\sigma^2) \sum_{i=1}^{k} (n_i - 1)s_i^2\) and by finding the critical point one sees that it is maximized by taking

\[
\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{k} (n_i - 1)s_i^2.
\]

When this is done

\[
L(\hat{\mu}_1, \ldots, \hat{\mu}_k, \hat{\sigma}^2, \ldots, \hat{\sigma}^2; z) \left[ \frac{1}{2\pi^{k/2} \prod_{i=1}^{k} (n_i - 1)s_i^2} \right]^{n/2} e^{-n/2}.
\]
Without restrictions on $\mu_i$ and $\sigma_i$ the maximum of $L$ is obtained by maximizing each term $L(\mu_i, \sigma_i)$ separately, and we know that the maximum likelihood estimates are given by $\hat{\mu}_i = \bar{x}_i$ and $\hat{\sigma}^2 = [(n_i - 1)/n_i]s_i^2$, and

$$L(\bar{x}_i, [(n_i - 1)/n_i]s_i^2; x_{i,1}, \ldots, x_{i,n_i}) = \left[\frac{1}{2\pi[(n_1 - 1)/n_i]s_i^2}\right]^{n_i/2} e^{-n_i/2}. \quad (2)$$

Thus

$$L(\hat{\mu}_1, \ldots, \hat{\mu}_k, \hat{\sigma}_1^2, \ldots, \hat{\sigma}_k^2; z) = \prod_{i=1}^k \left[\frac{1}{2\pi[(n_i - 1)/n_i]s_i^2}\right]^{n_i/2} e^{-n_i/2}. \quad (3)$$

Taking the ratio of the expressions in (2) and (3) leads to

$$\lambda = \frac{\prod_{i=1}^k [(n_i - 1)/n_i]s_i^2/n_i^{n_i/2}}{\left[\frac{1}{n} \sum_{i=1}^k (n_i - 1)s_i^2\right]^{n/2}}.$$