1. An urn contains $r = r_1 + r_2$ packages of candy; of these, $r_1$ contain one piece and $r_2$ contain three pieces.
(a) If a package were to be drawn from the urn at random, what is the probability it would contain one piece of candy? Three pieces? If two packages were drawn (without replacement), what are the probabilities of obtaining altogether two, four, or six pieces of candy?

For (b) and (c) below, we suppose that a sample of $n$ packages ($n \leq r$) is drawn from the urn, one at a time, without replacement.

(b) Let $X_i$ be the number of pieces of candy in the $i^{th}$ package drawn. Find $E[X_i]$, $\text{Var}(X_i)$, and $\text{Cov}(X_i, X_j)$ for $i \neq j$. Hint: use your results from (a).

(c) Let $X$ be the total number of pieces of candy obtained. Find $E[X]$ and $\text{Var}(X)$. Check your answer by inspection when $n = r$.

2. Let $X$ and $Y$ be continuous random variables with joint density

$$f(x, y) = \begin{cases} cx^2 ye^{-xy}, & \text{if } 1 \leq x \leq 2 \text{ and } y \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

(a) If we know that $X$ takes value $x$, what is the (conditional) distribution of $Y$? No calculations are needed: this is a question of recognizing a standard distribution.

Hint for (b)-(c): Set up as double integrals and do the $y$ integral first.

(b) Find the constant $c$.

(c) Find $E[X]$, $E[Y]$, and $\text{Cov}(X, Y)$.

3. (a) Xerxes and Yvonne play a game in which each, independently, chooses a real number between 0 and 2; their choices are called $X$ and $Y$, respectively. Xerxes wins if his choice is at least twice Yvonne’s. Suppose that both choose “at random”, so that their choices are uniformly distributed on the interval $[0, 2]$. What is the probability that Xerxes wins?

(b) Suppose that each time the game is played Yvonne contributes $2.00 and Xerxes $1.00 to a pot, with the winner of the game collecting the entire $3.00, and that they play the game 100 times in this fashion. Use the central limit theorem to estimate the probability that Yvonne comes out at least $35.00 ahead. (The half-integer correction would be a bit subtle here; don’t worry about using it.)

4. Alice and Bob play a game with a coin which shows heads with probability $p$. Alice flips the coin three times, and then Bob does also. The one with the most heads (out of his or her three flips) wins.

(a) What is the probability that the two players have a total of at least five heads?

(b) What is the probability that a tie occurs?

(c) What is the probability that Bob has already won the game after his first flip?

5. Three urns are numbered 1 through 3; urn $k$ contains $k$ balls numbered 1 through $k$. We select an urn at random, draw a ball from it, note the number of the ball, replace the ball, and then draw again from the same urn. If it is known that the first ball drawn has number 1, find (a) the probability mass function of the number of the selected urn; (b) the expected value of the number of the second ball drawn.

6. Let $X$ and $Y$ be independent continuous random variables with densities

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise}; \end{cases} \quad f_Y(y) = \begin{cases} 3e^{-3y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Find the density function $f_Z(z)$ for $Z = X + Y$ (a) by computing $F_Z(z)$; (b) using (3.2) of Chapter 6.

7. Let $X_1, X_2, \ldots, X_{25}$ be independent random variables, each of which is uniformly distributed on the interval $[0, 2]$, and let $X = \sum_{i=1}^{25} X_i$.

(a) Find the mean and variance of $X$.

(b) Use the central limit theorem to estimate the probability that $|X - 24| \leq 3$.

(c) Use Chebyshev’s inequality to find a number $a$ such that you are absolutely sure that $P\{|X - E[X]| \geq a\} \leq 0.1$. 

Find the density function $f_Z(z)$ for $Z = X + Y$ (a) by computing $F_Z(z)$; (b) using (3.2) of Chapter 6.
8. A single die is rolled; let us denote the number shown by $X$. Then a (possibly) biased coin, with probability $X/6$ of showing heads, is flipped; the number of heads (0 or 1) obtained is denoted $Y$. (For example, if a 3 is rolled then a fair coin is flipped.)

(a) Find the joint probability mass function, and the marginal probability mass functions, of $X$ and $Y$.
(b) Find $E[X]$, $E[Y]$, $Var(X)$, $Var(Y)$, and $Cov(X,Y)$.
(c) Find the probability mass function of the random variable $Z = X + Y$.

9. A modified form of poker is played with an ordinary deck of 52 playing cards, but hands contain six cards. Calculate the probability that a randomly dealt hand contains: (i) four cards of the same rank and two from different ranks; (ii) three pairs, each of a different rank; (iii) two triples, each containing three cards of the same rank; (iv) only two ranks; (v) six cards in sequence; (vi) six cards, all of different ranks; (vii) a triple, a pair, and a single card, all of different ranks.

10. (a) A hat contains $n$ cards numbered 1 to $n$; one card is drawn at random and its number is denoted $X$. Calculate the moment generating function $M_X(t)$ of $X$. (Hint: review the formula for the sum of a finite geometric series.)
(b) If the experiment is repeated $k$ times (with replacement) and $Y$ is the sum of the numbers of all cards drawn, find $M_Y(t)$.

11. (a) State Markov’s inequality as a theorem, defining all symbols used and including all necessary hypotheses.
(b) State Chebyshev’s inequality as a theorem, defining all symbols used and including all necessary hypotheses.

12. A large urn contains $N$ balls of each of 20 different colors (that is, a total of $20N$ balls). 10 balls are selected at random; we let $X$ be the total number of different colors obtained, and write $X = \sum_{i=1}^{20} X_i$ with $X_i$ a Bernoulli random variable indicating whether or not the $i^{th}$ color is obtained.

(a) Suppose that the selection is without replacement. Find $E[X]$ and $Var(X)$. (Hint: compute $E[X_i]$ by computing $P\{X_i = 0\}$ using equally likely outcomes; do $E[X_iX_j]$ similarly.)
(b) Find by elementary reasoning the correct values in (a) when $N = 1$, and verify that the answers you obtained there are correct in that case.
(c) Suppose that the selection is with replacement. Find $E[X]$ and $Var(X)$. Your answer should not depend on $N$.
(d) Show that your answers in (c) are the $N \to \infty$ limit of your answers in (a). Explain why this should be true.

13. At the beginning of a murder investigation, Inspector Hotchkiss always believes that there is a 75% chance that the butler did it. On the other hand, he also believes that there is a probability of 50% that a guilty person will react strongly when confronted with the body, but that the corresponding probability for an innocent person is only 10%. If the butler in the Pringle case views the body with no signs of emotion, what probability should the inspector then assign to his guilt?

14. Let $X$ and $Y$ be continuous random variables with joint density

$$f(x,y) = \begin{cases} 2xy, & \text{if } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq x/2, \\ 0, & \text{otherwise}. \end{cases}$$

Find $P\{X + Y \leq 2\}$.

15. Rose and Bob play a game in which each in turn draws a ball from the urn containing one red and four blue balls, and then replaces it; Rose draws first.

(a) Suppose that Rose wins if she draws a red ball before Bob draws a blue ball, and Bob wins if he draws a blue ball before Rose draws a red ball. What is the probability that Rose wins? Hint: condition on the results of the first two draws.
(b) If the rules are modified so that Rose wins if she draws a red ball before Bob draws a blue ball twice, what then is the probability that Rose wins?

16. Discuss one of the following classical problems in probability, describing the problem (or model, or situation) and giving some details about the solution: gambler’s ruin, the problem of the points, coupon collecting, the matching problem, Buffon’s needle.