PRACTICE COMBINATORIAL THEORY QUESTIONS

Question 1. Suppose that on a street with 20 houses, 7 have a person ill with the flu.

(a) In how many ways can the presence or absence of flu occur so that all of these 7 houses are next to each other?

(b) In how many ways can the presence or absence of flu occur so that none of these 7 houses are next to each other?

The answers are:

(a) 14
(b) \(2 \times \left(\frac{13!}{7!6!}\right)\)

Question 2. Solve the recurrence

\[a_{k+1} = 5a_k + 7^k, \quad k \geq 1\]

\[a_1 = 6.\]

The answer is:

\[a_k = \frac{5^k + 7^k}{2}\]

Question 3. Solve the recurrence

\[a_{n+1} = (n + 1)a_n + 7, \quad n \geq 0\]

\[a_0 = 7.\]

The answer is:

\[a_n = 7n! \left(1 + 1/1! + 1/2! + \cdots + 1/n!\right)\]
Question 4.

(a) Use the principle of inclusion/exclusion to compute the chromatic polynomial of the following graph.

(b) Find the number of distinguishable ways in which the letters $a, a, b, b, c, c, d, d, d$ can be arranged so that two letters of the same kind never appear consecutively.

(c) Find the number of distinguishable ways in which the letters $a, a, b, b, c, c, d, d, d, d$ can be arranged so that there are at least two consecutive letters of type $d$.

The answers are:

(a) $x^5 - 5x^4 + 10x^3 - 9x^2 + 3x$

(b) $P(8; 2, 2, 2, 2) - 4P(7; 1, 2, 2, 2) + 6P(6; 1, 1, 2, 2) - 4P(5; 1, 1, 1, 2) + P(4; 1, 1, 1, 1)$

(c) $2P(8; 2, 2, 2) - P(7; 2, 2, 2, 1)$

Question 5.

(a) Define the chromatic number of the finite graph $G$.

(b) Show that $p(x) = x^5 - 9x^4 + 22x^3 - 18x^2 + 4x$ is not the chromatic polynomial of any graph.

(c) Find a graph $G$ such that

$$P(G, x) = x^7 - 5x^6 + 9x^5 - 7x^4 + 2x^3.$$
Question 6. A word from the alphabet \{0, 1, 2, 3\} is legitimate if no two 0’s appear consecutively.

(a) Find a recurrence for the number \(b_n\) of legitimate words of length \(n\).

(b) Solve the recurrence.

(c) A word from the alphabet \{0, 1, 2, 3\} is valid if
   
   (i) 0 and 2 occur an even number of times, and
   
   (ii) 1 and 3 occur an odd number of times.

   Find the number \(c_n\) of valid words of length \(n\).

The answers are:

(a) \(b_{n+1} = 3b_n + 3b_{n-1}\), where \(b_1 = 4\) and \(b_2 = 15\).

(b) The solution of the recurrence is

\[
b_n = \left(\frac{\sqrt{21} + 5}{2\sqrt{21}}\right) \left(\frac{3 + \sqrt{21}}{2}\right)^n + \left(\frac{\sqrt{21} - 5}{2\sqrt{21}}\right) \left(\frac{3 - \sqrt{21}}{2}\right)^n
\]

(c) For \(n \geq 1\), the required number is

\[
c_n = \frac{4^n + (-4)^n}{16}
\]

Question 7.

(a) Give the definition of a Latin square of order \(n\).

(b) Give the definition of when two Latin squares of order \(n\) are orthogonal.

(c) Construct a complete family of orthogonal Latin squares of order 5.

(d) Give an example of a pair of orthogonal Latin squares of order 5, both of which have first row

\[
1 \ 5 \ 3 \ 4 \ 2
\]