1. Section 17:8 Solution. Part a. $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$ is always true, but $\bar{A} \cap \bar{B} \subset \overline{A \cap B}$ need not hold.

Proof of $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$: We have $A \cap B \subset A \subset \bar{A}$ and $A \cap B \subset B \subset \bar{B}$ so $\overline{A \cap B}$ is a closed set containing $A \cap B$. By definition of $\overline{A \cap B}$ we must have $\overline{A \cap B} \subset C = \bar{A} \cap \bar{B}$.

Counterexample to $\bar{A} \cap \bar{B} \subset \overline{A \cap B}$. Take the standard topology on $\mathbb{R}$ and let $A = (0, 1)$ and $B = (1, 2)$. Then $A \cap B = (0, 1] \cap [1, 2) = \{1\}$ but $\overline{A \cap B} = [0, 1] \neq \emptyset$.

Part b. Similar to part a, one containment always holds the other may not hold. The counterexample in part a is also a counterexample for part b. The proof in part a is easily generalized.

Part c. $\overline{A-B} \subset \bar{A} - \bar{B}$ need not hold. Consider the example in the standard topology on $\mathbb{R}$: $A = [0, 2]$ and $B = [1, 2]$ then $A - B = [0, 1)$ and $\overline{A-B} = [0, 1]$ but $\bar{A} - \bar{B} = A - B = [0, 1)$. $\overline{A-B} \subset \bar{A} - \bar{B}$ always holds. Suppose $x \in \overline{A-B}$, we will show $x \in \bar{A} - \bar{B}$. Let $N$ be an arbitrary neighborhood of $x$. We must show $N \cap (A - B) \neq \emptyset$. Since $x \notin \bar{B}$, there is a neighborhood $M$ of $x$ with $M \cap B = \emptyset$. Then $M \cap N$ is a neighborhood of $x$ and so $x \in \bar{A}$ implies $M \cap N \cap A \neq \emptyset$. Let $z \in M \cap N \cap A$. Then $z \in A \cap M \subset A - B$, since $M \cap B = \emptyset$, as required.

2. Section 17:13 Solution. Let $(X, T)$ be a topological space. First we assume $(X, T)$ is Hausdorff and show that $\Delta = \{(x, x) : x \in X\}$ is closed in the product $(X^2, U) = (X, T) \times (X, T)$. To show $\Delta$ is closed, we show $X^2 - \Delta$ is open. For this we must show that any point of $X^2 - \Delta$ has a neighborhood that is disjoint from $\Delta$. Let $(y, z)$ be an arbitrary point in $X^2 - \Delta$, where $y$ and $z$ are each in $X$. Since $(y, z) \notin \Delta$, $y \neq z$. Since $(X, T)$ is a Hausdorff space, there are neighborhoods $M$ of $x$ and $N$ of $y$ such that $M \cap N = \emptyset$. By the definition of the product topology, $M \times N$ is a neighborhood of $(y, z)$ in $(X^2, U)$ and since $M \cap N = \emptyset$, $(M \times N) \cap \Delta = \emptyset$, as required.

Now assume $\Delta$ is closed in $(X^2, U)$. We must show that $(X, T)$ is a Hausdorff space. Let $y, z$ be distinct points of $X$. Then $(y, z) \notin \Delta$ and since $\Delta$ is closed, there is a neighborhood $U$ of $(y, z)$ in $(X^2, U)$ that is disjoint from $\Delta$. Since $U$ is a neighborhood of $(y, z)$ in the product topology of $(X, T)$ with itself, we can find open sets $M, N$ in $(X, T)$ such that $(y, z) \in M \times N \subset U$. Then $M$ is a neighborhood of $x$ and $N$ is a neighborhood of $y$ in $(X, T)$. We claim $M \cap N = \emptyset$. Suppose not, and let $w \in M \cap N$. Then $(w, w) \in M \times N \subset U$ contradicting that $U \cap \Delta = \emptyset$. So $M$ is a neighborhood of $x$ and $N$ is a neighborhood of $y$ and $M \cap N = \emptyset$ as required to show that $(X, T)$ is a Hausdorff space.

3. Section 17:16 or 17:17

Solution to 17:16. You are asked to find the closure of the set $K = \{1/n : n \in \mathbb{Z}_+\}$ in each of five topologies.

\[^1\text{Version: 10/14/10}\]
Part b.

- \( T_1 \), the standard topology. The closure is \( K \cup \{0\} \). We must show: (i) \( 0 \in \bar{K} \) and (ii) \( K \cup \{0\} \) is closed. To show (i), let \((a, b)\) be an open interval containing 0, we must show \((a, b) \cap K \neq \emptyset \). We must have \( a < 0 < b \), so there is an integer \( n \) so that \( 1/n < b \), so \( 1/n \in (a, b) \cap K \). To show (ii), we show that \( \mathbb{R} - (K \cup \{0\}) \) is open. Now \( \mathbb{R} - (K \cup \{0\}) \) is equal to \((\infty, 0) \cup (1, \infty) \cup \bigcup_{n \in \mathbb{Z}_+}(1/(n+1), 1/n)\) which is a union of open sets in the standard topology and therefore is open.

- \( T_2 \), the topology of \( \mathbb{R}_K \). \( K \) is closed in this topology since \( \mathbb{R} - K \) is open, since \( \mathbb{R} - K \) can be written as a union of all open sets of the form \((-r, r) - K\), where \( r > 0 \).

- \( T_3 \), the finite complement topology. Here \( \bar{K} = \mathbb{R} \). To show this we need to show that for every \( x \in \mathbb{R}, x \in \bar{K} \). Let \( x \in \mathbb{R} \) be arbitrary and let \( N \) be an arbitrary neighborhood of \( x \). Then \( N = \mathbb{R} - F \) where \( F \) is a finite set. Since \( K \) is infinite, \( K \) can’t be a subset of \( F \) so \( K \cap N \neq \emptyset \), showing \( x \in \bar{K} \).

- \( T_4 \), the upper limit topology. Here \( K \) is closed so \( \bar{K} = K \). To show this, we show that \( \mathbb{R} - K \) is open. We can write \( \mathbb{R} - K \) as \((-\infty, 0] \cup (1, \infty) \cup \bigcup_{n \in \mathbb{Z}_+}(1/(n+1), 1/n)\). We need to show that each of the sets in this union is open in the upper limit topology. We will do this by showing that each one is a union of basis sets for the upper limit topology. We have that \((-\infty, 0] = \bigcup_{n \in \mathbb{Z}_+}(-n, 0]\) and \((1, \infty) = \bigcup_{r \geq 1}(1, r]\). Any interval of the form \((a, b)\) can be written as \( \bigcup_{x \in (a, b]}(a, r]\), so each of the sets \((1/(n+1), 1/n)\) is also open. Therefore \( \mathbb{R} - K \) is open.

- \( T_5 \), the topology having as basis all sets \((-\infty, a)\). Here \( \bar{K} \) is the set of nonnegative real numbers. Let \( x \in \mathbb{R} \), we must show that if \( x \geq 0 \) then \( x \in \bar{K} \) and if \( x < 0 \) then \( x \notin \bar{K} \). Suppose \( x \geq 0 \). If \((\infty, a)\) is a neighborhood of \( x \) then we must have \( a > 0 \) so there is a positive integer \( n \) with \( 1/n < a \) so \( 1/n \notin (\infty, a) \cap K \), showing \( a \notin \bar{K} \). If \( x < 0 \) then \( x \) is in the open set \((-\infty, 0)\) which is disjoint from \( K \) so \( x \notin \bar{K} \).

Part b.

- \( T_1 \) is Hausdorff, since if \( x < y \) we choose \( \varepsilon = (y - x)/3 \) and then \((x - \varepsilon, x + \varepsilon)\) and \((y - \varepsilon, y + \varepsilon)\) are neighborhoods of \( x \) and \( y \) that are disjoint. It is \( T_1 \) since Hausdorff implies \( T_1 \).

- \( T_2 \) is Hausdorff using the same argument as for \( T_1 \) (since \( T_2 \) is finer than \( T_1 \)).

- \( T_3 \) is not Hausdorff. To see this, suppose for contradiction that it is. Then there are neighborhoods \( N_0 \) of 0 and \( N_1 \) of 1 that are disjoint so \( N_1 \subset \mathbb{R} - N_0 \). By the definition of the topology \( \mathbb{R} - N_0 \) is a finite set, so \( N_1 \) must also be a finite set. But \( \mathbb{R} - N_1 \) is also a finite set, so \( \mathbb{R} = N_1 \cup (\mathbb{R} - N_1) \) is the union of two finite sets \( N_1 \) and \( \mathbb{R} - N_1 \), a contradiction.

- \( T_3 \) is \( T_1 \) since for \( x \in \mathbb{R}, \mathbb{R} - \{x\} \) is open, so \( \{x\} \) is closed.

- \( T_4 \) is Hausdorff and \( T_1 \): Proof is similar to the one for \( T_1 \).

- \( T_5 \) is not Hausdorff nor \( T_1 \). It’s enough to show that it’s not \( T_1 \), since not \( T_1 \) implies not Hausdorff. Every closed set containing \( \{0\} \) has the form \([-a, \infty)\) where \( a \leq 0 \), so the closure of \( \{0\} \) is the set of all nonnegative real numbers.

Solution to 17:17.
• Closure of $I = (0, \sqrt{2})$ in lower limit topology, where open sets are all sets of the form $[a, b)$ where $a, b \in \mathbb{R}$. The closure is $[0, \sqrt{2}] = I \cup \{0\}$. We need to show that (1) Every point of $I \cup \{0\}$ is in the closure of $I$ and (2) $I \cup \{0\}$ is closed. For (1), obviously the closure contains $I$. To see $0$ is in the closure, take any neighborhood of $0$, which must have the form $[a, b)$ where $a \leq 0 < b$, so $b/2 \in [a, b) \cap I$ and so the neighborhood has a point in common with $I$ and so $0 \in \bar{I}$. To prove (2), note that $\mathbb{R} - [0, \sqrt{2}] = (-\infty, 0) \cup [\sqrt{2}, \infty)$ is open in this topology since it is the union of the open sets $[\sqrt{2}, \infty)$ and $[-n, 0)$ for all $n \in \mathbb{Z}_+$. 

• Closure of $I = (0, \sqrt{2})$ in the topology of exercise 13:8b where open sets are all sets of the form $[a, b)$ where $a, b$ are rational. The closure is $[0, \sqrt{2}] = I \cup \{0, \sqrt{2}\}$. The argument that $0 \in \bar{I}$ is the same as for the lower limit topology. To see $\sqrt{2} \in \bar{I}$, let $N$ be a neighborhood of $\sqrt{2}$. So there is a basis set $[a, b) \subset N$ with $a, b$ rational and $a \leq \sqrt{2} < b$. Since $\sqrt{2}$ is not rational, we have $a < \sqrt{2} < b$. Therefore $\bar{I} \cap [a, b) = [a, \sqrt{2})$ is nonempty, establishing that $\sqrt{2} \in \bar{I}$.

Next we prove that $[0, \sqrt{2}]$, by showing its complement $(-\infty, 0) \cup (\sqrt{2}, \infty)$ is a union of open sets and hence open. We have $(-\infty, 0) = \bigcup_{n \in \mathbb{Z}_+} [-n, 0)$ and $(\sqrt{2}, \infty)$ is the union of all interval $[c, d)$ where $c$ and $d$ are rational numbers satisfying $\sqrt{2} < c < d$.

• Closure of $J = (\sqrt{2}, 3)$ in lower limit topology, is $[\sqrt{2}, 3) = J \cup \{\sqrt{2}\}$. The proof of this follows a similar argument to the proof that the closure of $I$ was $I \cup \{0\}$.

• Closure of $J = (\sqrt{2}, 3)$ in the topology of exercise 13:8b. Here the closure is $[\sqrt{2}, 3) = J \cup \{\sqrt{2}\}$ (note that $3$ is not in the closure). The proof that $\sqrt{2}$ is in the closure is (again) similar to the proof that $0$ is in the closure of $I$ in the lower limit topology. To prove that $J \cup \sqrt{2} = [\sqrt{2}, 3)$ is closed write it’s complement $(-\infty, \sqrt{2}) \cup [3, \infty)$ as a union of open sets. $(-\infty, \sqrt{2})$ is the union of the open sets $(a, b)$ where $a, b$ are rational and $a < b < \sqrt{2}$. $[3, \infty)$ is the union of the open sets $[3, c)$ where $c$ is a rational number greater than $3$.

### 4. Section 17: 19 a,c,d

**Solution.** The following lemma will simplify the writing of the proofs.

**Lemma.** For a topological space $(X, T)$ and $A \subset X$, $X - Int(A) = \overline{X - A}$. To see this observe that $Int(A) \subset A$ is an open set so $X - Int(A)$ is a closed set containing $X - A$ so $(X - A) \subset X - Int(A)$. On the other hand $X - \overline{X - A}$ is an open set contained in $A$ so $X - \overline{X - A}$ is an open set contained in $A$ so is a subset of $Int(A)$. Therefore $X - Int(A) \subset (X - A)$.

- (a). To show $Int(A) \cap Bd(A) = \emptyset$, the lemma tells us $(X - A) \subset X - Int(A)$. Since $Bd(A) \subset (X - A)$, we have $Bd(A) \subset X - Int(A)$, which means $Bd(A) \cap Int(A) = \emptyset$. To show $\overline{A} = Int(A) \cup Bd(A)$ we see first that $Int(A) \subset A \subset \overline{A}$ and $Bd(A) \subset \overline{A}$ (by definition of $Bd(A)$) so $Int(A) \cup Bd(A) \subset \overline{A}$. Now we show $\overline{A} \subset Int(A) \cup Bd(A)$. Let $x \in \overline{A}$. If $x \in Int(A)$ we’re done, so assume $x \not\in Int(A)$. Then $x \in X - Int(A)$ which by the lemma is equal to $(X - A)$. Then $x \in \overline{A} \cap (X - A) = Bd(A)$.

- (c) First assume $U$ is open. Then $X - U$ is closed so $Bd(U) = \overline{U} \cap (X - U) = \overline{U} - U$. Next assume $Bd(U) = \overline{U} - U$; we want to show that $U$ is open which is the same as showing that $X - U$ is closed. Let $z \in U$, we need to show that $z \not\in (X - U)$. Since
$z \in U$, $z \notin \bar{U} - U = Bd(U)$. By definition, $Bd(U) = \bar{U} \cap (X - U)$. Since $z \in U \subset \bar{U}$ but not in $Bd(U)$ it must be that $z \notin (X - U)$, as required to show that $X - U$ is closed and $U$ is open.

- (d) It is possible that $U$ is open and $U \neq Int(\bar{U})$. Consider the standard topology on $\mathbb{R}$. The set $U = (0,1) \cup (1,2)$ has $\bar{U} = [0,2]$ and $Int(\bar{U}) = (0,2)$.