1. (a) Give a clear definition of the product of two topological spaces.
   (b) Prove that for two spaces \((X, T)\) and \((Y, U)\) that the product of a closed subset of \(X\) with a closed subset of \(Y\) is closed in the product topology.

2. (a) Give a clear definition of what it means for a topological space to be Hausdorff.
   (b) Give a clear definition of what it means for a sequence of points in a topological space to converge to a limit \(L\).
   (c) Prove that in a Hausdorff space a sequence can converge to at most one limit.

3. (a) Give a clear definition of what it means for one topology to be strictly finer than another.
   (b) Let \((\mathbb{R}, T_1)\) denote the standard topology on \(\mathbb{R}\), and let \((\mathbb{R}, T_2)\) denote the lower limit topology, whose basis consists of sets \([a, b) : a < b\}. Prove that \(T_2\) is strictly finer than \(T_1\).

4. Let \((X, T_1)\) and \((X, T_2)\) be two spaces and for \(A \subseteq X\) let \(cl_1(A)\) be the closure of \(A\) in the space \((X, T_1)\) and let \(cl_2(A)\) be the closure of \(A\) in the space \((X, T_2)\). Prove that if \(T_1\) is finer than \(T_2\) then \(cl_1(A) \subset cl_2(A)\).

5. Consider the set \([0,1] \times [0,1]\) with the dictionary order topology. Determine (with proof) the closure of the set \((0,1) \times (0,1)\).

6. Let \((X_1, T_1)\) and \((X_2, T_2)\) be topological spaces and let \((X_1 \times X_2, U)\) be the product space. Define the function \(\pi_1 : X_1 \times X_2 \rightarrow X_1\) by the rule \(\pi_1((x_1, x_2)) = x_1\). Prove that \(\pi_1\) maps any open set of \((X_1 \times X_2, U)\) to an open set of \((X_1, T_1)\).

7. (a) Define what is meant by the notion of “immediate predecessor” in an ordered set \((X, <)\).
   (b) (Corrected 10-18-10) Let \((X, <)\) be an ordered set and \((X, T)\) be the order topology. Prove that for each \(a \in X\), the set \(\{x \in X : x \geq a\}\) is open if and only if \(a\) has an immediate predecessor or is the minimum element.

8. (a) Define what is meant by the interior of a set in a topological space.
   (b) Prove: for any set \(A\) in a topological space \((X, T)\), the complement of the closure of \(A\) is equal to the interior of the complement of \(A\).

9. Prove that every well-ordered set has the least upper bound property.

10. A point \(x\) of a topological space is ordinary if the intersection of all open sets containing \(x\) is equal to \(\{x\}\). Prove: In any topological space, if every singleton set is closed, then every point is ordinary.

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