(1) Section 4: 9

Comments.

- The purpose of this problem was to get some practice working only with the axioms of the real numbers and integers. The axioms do not include things like: decimal expansion of real numbers, the floor and ceiling function, etc. which would make this problem easy. Part b of the problem can be viewed as providing a proof that the floor and ceiling functions exist.

- In part a, the hypothesis we have a “nonempty subset of \( \mathbb{Z} \) that is bounded above”. The hypothesis does not say that the upper bound is an integer. It may be a real number. This makes the problem a bit harder. You have to use the Archimedean principle to say that if a set is bounded above by a real number then it is bounded above by an integer.

- Here’s a sketch of the proof for part b: Split into two cases: \( x > 0 \) and \( x < 0 \). Case 1. Assume \( x > 0 \). Let \( S \) be the set of integers that are greater than \( x \). \( S \) is nonempty (by the Archimedean principle) and consists of positive integers so by the well-ordering of \( \mathbb{Z}_+ \), \( S \) has a minimum \( m \). Let \( n = m - 1 \). Then \( n < x \) since \( n + 1 \) is the minimum of \( S \) and \( x < n + 1 \). Case 2. Assume \( x < 0 \). Then \(-x\) is a positive real number so by the first case there is an integer \( t \) so that \( t < -x < t + 1 \) and so if \( n = -(t+1) \) we have \( n < x < n + 1 \).

(2) Section 7: 4. No comments.

(3) Section 10: 6

Comments.

(a) The difficulty in this problem is that you are dealing with this funny ordered set which has strange properties. Many of you made assumptions that are not valid.

(b) For example, a number of students assumed the following: Any ordered set in which every element has an immediate predecessor and an immediate successor must be countable. This is not true. Consider the following example: Take the dictionary order on \( \mathbb{R} \times \mathbb{Z} \). Every element in this order has both an immediate predecessor and an immediate successor.

(c) Part a is easy. Hint: Suppose for contradiction \( S_\Omega \) has a largest element \( z \). Use this to show that \( S_\Omega \) is countable which will be a contradiction.

(d) Here is a false proof of part a: If \( S_\Omega \) had a maximum element \( m \) then \( S_\Omega - \{ m \} \) would be a proper uncountable subset of \( S_\Omega \) which contradicts that \( S_\Omega \) is the minimal uncountable well-ordered set. The error here is that the term “minimal uncountable well-ordered set” is defined by the author to mean an uncountable well-ordered set with the property that every section is countable, and so the above argument misuses the definition.

(e) Part b is also easy. Hint: Notice that for any element \( \alpha \) \( S_\Omega \) is the union of the sets \( S_\alpha \), \( \{ \alpha \} \) and \( \{ x : \alpha < x \} \).

(f) Here is an example of something strange that shows why part c is somewhat tricky. Let \( Pre \) be the set of elements that have an immediate predecessor and \( Suc \) be the set of elements that have an immediate successor. There is a bijection between \( Pre \) and \( Suc \) since if \( x \in Pre \) then its immediate predecessor \( y \) is in \( Suc \) and \( x \) is the immediate successor of \( y \). However, in any well-ordered set every element with the exception of the maximum element if there is one, belongs to \( Suc \), but there could be uncountably many elements that don’t belong to \( Pre \). For example, consider the example \( \mathbb{R} \times \mathbb{N} \).
with the dictionary order. All elements have an immediate successor, but there are uncountably many elements with no immediate predecessor. (Which are they?)

(g) There are a few ways to prove part c. The easiest way uses Theorem 10.3 which says that every countable subset of \( S_\Omega \) has an upper bound. Suppose for contradiction that \( X_0 \) is countable. Since \( X_0 \) is countable it has an upper bound, which we’ll call \( z_0 \). Now define a sequence of elements \( z_1, z_2, \ldots \) inductively by defining \( z_{i+1} \) to be the minimum element among all elements greater than \( z_i \). (This minimum exists because by part (b) the set of elements greater than \( z_i \) is nonempty.) Now let \( B_0 \) be the set of elements bigger than \( z_0 \) and let \( W = B_0 - \{ z_i : i \in \mathbb{N} \} \). Then \( W \) is nonempty (since \( B_0 \) is uncountable and \( \{ z_i : i \in \mathbb{N} \} \) is countable) so it has a minimum element \( w \). We claim that \( w \) has no immediate predecessor, which implies \( w \in X_0 \), and contradicts that \( z_0 \) is an upper bound on \( X_0 \). If \( w \) had an immediate predecessor \( p \) then we would have \( p > z_i \) for all \( i \) contradicting that \( w \) is the minimum of \( W \).

(4) Let \( A \) be the set of all infinite sequences of nonnegative integers \( y = (y_1, y_2, \ldots) \) having the property that only a finite number of terms are nonzero. Define the relation \( R \) on \( A \) by \( xRy \) if there is a positive integer \( n \) such that \( x_n < y_n \) and for all \( i > n \), \( x_i = y_i \).

(a) Prove that \( R \) is a total ordering relation on \( A \).

Comments
(i) To show that \( x, y \in A \) are comparable many of you correctly let \( n \) be the lowest index such that \( x_j = 0 \) for all \( j \geq n \) and \( m \) be the lowest index such that \( y_j = 0 \) for all \( j \geq m \). Then divide into cases according to \( m = n, m < n \) or \( m > n \).
(ii) Some of you did something like the above that doesn’t work: Letting \( n \) be the number of nonzeros in \( x \) and \( m \) be the number of nonzeros in \( y \). The problem is that the nonzeros may not appear consecutively, so comparing \( n \) and \( m \) does not give the needed information to compare \( x \) and \( y \).

(b) Prove that for each positive integer \( n \) there is a section of \( A \) that has the same order type as \( (\mathbb{Z}_+)^n \) with the dictionary order.

Comments
(i) Some of you tried to prove that \( A \) has the same order type as \( (\mathbb{Z}_+)^n \). There are various problems with this statement. What is \( n \) here? Are you saying there is such a bijection for all \( n \)? Or are you saying there exists an \( n \) for which this is true? You must specify what you mean or else it does not make sense. But even if you did specify what you mean, neither of these is true.
(ii) What is true is that for each positive integer \( n \) there is a section of \( A \) that has the same order type as \( (\mathbb{Z}_+)^n \). The easiest way to prove this is: Start as usual for a “For all \( n \)” statement by letting \( n \) be an arbitrary positive integer. Then identify a specific element \( s_n \) of \( A \) such that the section \( A_n \) of \( s_n \) has the same order type as \( (\mathbb{Z}_+)^n \). The choice of \( s_n \) that works here is the sequence that is all 0 except for a 1 in position \( n + 1 \). Then you need to show that this has the same order type as \( (\mathbb{Z}_+)^n \) which you can do by defining an appropriate bijection between these sets and showing that it is order preserving. (The bijection that works requires a little thought since it requires reversing the order of the coordinates.)

(c) Prove that \( A \) with the given ordering is a well-ordering.

Comments.
(i) To prove this you need to let \( Y \) be an arbitrary nonempty subset of \( A \) and show that \( Y \) has a smallest element. There are a few ways to do this. One way, that uses part (b), is to let \( y \) be an arbitrary element of \( Y \). If \( y \) is the all 0 element then it is the smallest element of \( A \) and hence also of \( Y \). So assume \( y \) is not all 0, and let \( n \) be the largest nonzero entry of \( y \). Let \( s_n \) be the element defined
in part (b) (having 0 in all positions except a 1 in position $n + 1$). Let $A_n$ be the set of elements less than $s_n$. In part (b) it was shown that $A_n$ has the same order type as $(\mathbb{Z}_+)^n$ with the dictionary order, and so $A_n$ is well-ordered. Note that $y \in Y \cap A_n$ so $Y \cap A_n$ is nonempty and so has a smallest element $t$ and this element must be the smallest element of $Y$ since every element of $Y \cap A_n$ is less than every element of $Y - A_n$ (since $A_n$ is a section.)

(ii) Some of you wrote something like: By part (b), $A$ has the same order type as $(\mathbb{Z}_+)^n$ with the dictionary order, which is well-ordered, so $A$ is well-ordered. But that is not an accurate statement of what was proved in (b). What is $n$ here? Actually, the order type of $A$ is different from that of $(\mathbb{Z}_+)^n$ with the dictionary order for all $n$.

(iii) Some students argued as follows: $A$ is countable and is a total order and since $\mathbb{N}$ is a well-order so is $A$. This is invalid reasoning because $\mathbb{Q}$ with the usual order is a countable partial order that is not a well-order.