A note on our proof of the Urysohn Lemma

In this note we complete the checking of a point in our proof of the Urysohn Lemma, a point which was left as an exercise on November 24. Recall that X is a second countable regular (and hence normal) space, that A and B are disjoint closed subsets of X, and that we are constructing a continuous function $f : X \to [0,1]$ with $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. We introduced the *dyadic rationals* $\mathbb{D}$, the set of all rational number of the form $j/2^n$ for j and n integers, and wanted to define for each $r \in \mathbb{D}$ an open set $U_r \subset X$ so that

$$\overline{U}_r \subset U_s \qquad \text{if } r < s. \tag{1}$$

We wrote $\mathbb{D} = \bigcup_{n=0}^{\infty} \mathbb{D}_n$ (a disjoint union), where

$$\mathbb{D}_0 = \{r \in \mathbb{D} \mid r \leq 0 \text{ or } r \geq 1\} \quad \text{ and } \quad \mathbb{D}_n = \left\{ \frac{j}{2^n} \,\middle|\, 1 \leq j \leq 2^n - 1,\ j \text{ odd} \right\} \quad \text{ if } n \geq 1.$$

For $r \in \mathbb{D}_0$ we defined U_r so that (1) is satisfied for all $r, s \in \mathbb{D}_0$; specifically, we defined $U_r = \emptyset$ if $r < 0$, $U_r = X$ if $r > 1$, U_1 satisfying $A \subset U_1 \subset \overline{U}_1 \subset X - B$, and U_0 satisfying $B \subset U_0 \subset \overline{U}_0 \subset U_1$.

Next we defined U_r for all $r \in \mathbb{D}$ by recursion, as follows. Let $C_n = \bigcup_{k=0}^{n} \mathbb{D}_k$. We supposed inductively that we had defined U_r for $r \in C_n$ in such a way that (1) was satisfied for all $r, s \in C_n$; this induction assumption held for $n = 0$. Then if $r = (2j+1)/2^{n+1} \in \mathbb{D}_{n+1}$ we defined U_r to be an open subset of X satisfying

$$\overline{U}_{j/2^n} \subset U_r \subset \overline{U}_r \subset U_{(j+1)/2^n}; \tag{2}$$

we could do so because X is normal. It was then necessary to check that (1) is satisfied for all $r, s \in \bigcup_{k=0}^{n+1} \mathbb{D}_k$. It is this point which was left as an exercise and which we now take up.

So suppose that $r, s \in \bigcup_{k=0}^{n+1} \mathbb{D}_k$ with, say, $r < s$. If both r and s lie in C_n then (1) holds by our inductive assumption. If $r \in \mathbb{D}_{n+1}$, say $r = (2j+1)/2^{n+1}$, and $s \in C_n$ then $r < (j+1)/2^n \leq s$ and so

$$\overline{U}_r \subset U_{(j+1)/2^n} \subset \overline{U}_{(j+1)/2^n} \subset U_s,$$

where the first inclusion comes from (2) and the last from the induction assumption, since both $(j+1)/2^n$ and s lie in C_n. The verification of (1) when $r \in C_n$ and $s \in \mathbb{D}_{n+1}$ is similar. Finally, if $r = (2j+1)/2^{n+1} \in \mathbb{D}_{n+1}$ and $s = (2k+1)/2^{n+1} \in \mathbb{D}_{n+1}$, with $j < k$, then $j + 1 \leq k$ and so

$$\overline{U}_r \subset U_{(j+1)/2^n} \subset U_{k/2^n} \subset \overline{U}_{k/2^n} \subset U_s,$$

where again the first and last inclusions are from (2) and the second inclusion is immediate if $j+1 = k$ and follows from the inductive assumption, via $U_{(j+1)/2^n} \subset \overline{U}_{(j+1)/2^n} \subset U_{k/2^n}$, if $j + 1 < k$. This completes the recursive verification of (1). ■