THE RIEMANN-LEBESGUE LEMMA

Strauss avoids proving this result in his Ch. 5, but that obliges him to use an ad hoc dodge to establish the special case that he needs to prove his pointwise convergence theorem for Fourier series. My feeling is that since the Riemann-Lebesgue lemma (for Riemann-integrable functions) is so easy to prove and is useful in so many situations, we might as well take the time to give the proof.

0. Statement of the Riemann-Lebesgue Lemma: The result is easy to state:

Let \( f \) be a Riemann-integrable function defined on an interval \( a \leq x \leq b \) of the real line. Then for any real \( \beta \)

\[
\lim_{|\alpha| \to \infty} \int_a^b f(x) \cos(\alpha x + \beta) \, dx = 0.
\]

This statement is less general than it looks. Because \( \cos(\alpha x + \beta) = \cos \beta \cos(\alpha x) - \sin \beta \sin(\alpha x) \), the statement of the lemma is in fact equivalent (take \( \beta \) to be a suitable multiple of \( \pi / 2 \)) to the two statements

\[
\lim_{|\alpha| \to \infty} \int_a^b f(x) \cos(\alpha x) \, dx = 0 \quad \text{and} \quad \lim_{|\alpha| \to \infty} \int_a^b f(x) \sin(\alpha x) \, dx = 0,
\]

or to the complex form

\[
\lim_{|\alpha| \to \infty} \int_a^b f(x) e^{i \alpha x} \, dx = 0
\]

and it’s a toss-up as to which is the easiest to prove (though I think it’s the complex form). In any event, the proof proceeds by cases, moving from very simple functions to general Riemann-integrable functions. I’ll only state the proof for the cosine, but it’s easy to see how to change it to the case of the sine or to the complex-exponential formulation.

1. The Constant Case: This is absurdly simple: if \( f(x) \equiv c \) is a constant function, then

\[
\int_a^b c \cos(\alpha x + \beta) \, dx = c \cdot \frac{\sin(\alpha b + \beta) - \sin(\alpha a + \beta)}{\alpha}
\]

\[
\left| \int_a^b c \cos(\alpha x + \beta) \, dx \right| \leq |c| \cdot \frac{1 + 1}{|\alpha|} \to 0 \quad \text{as} \quad |\alpha| \to \infty.
\]

2. The Step-Function Case: Here we need to recall what a step function on the interval \([a, b]\) is, namely, a function defined in the following way. One takes a finite “partition” \( a = x_0 < x_1 < \cdots < x_n = b \) of the interval \([a, b]\), a sequence of \( n \) constants \( c_1, \ldots, c_n \), and defines the function \( s(x) \) by

For \( a \leq x \leq b \), \( s(x) = c_j \) if \( x_{j-1} \leq x < x_j \) (\( j = 1, \ldots, n \))

\( s(b) = c \) (this value can be chosen arbitrarily).

In the setting of the Riemann-Lebesgue lemma, a function defined in this way behaves just as constants do, because it is “piecewise constant”:

\[
\int_a^b s(x) \cos(\alpha x + \beta) \, dx = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} c_j \cdot \cos(\alpha x + \beta) \, dx
\]

\[
\left| \int_a^b s(x) \cos(\alpha x + \beta) \, dx \right| \leq \sum_{j=1}^{n} |c_j| \cdot \frac{2}{|\alpha|} \to 0 \quad \text{as} \quad |\alpha| \to \infty, \text{ term-by-term}.
\]
3. Riemann-Integrable Functions: Because of the definition of the Riemann integral, this case isn’t much different from that of step functions; however, it’s one of the few places in elementary mathematics at which one has to go back to the definition of the Riemann integral in order to prove something.\(^{(1)}\) One of the ways of defining the notion “\(f(x)\) is Riemann-integrable on \([a,b]\)” is to say\(^{(2)}\) that “the result of approximating it by circumscribed rectangles.” Stated formally, this would say that for every preassigned error \(\epsilon > 0\) it is possible to find a partition \(a = x_0 < x_1 < \cdots < x_n = b\) of the interval and two sequences of numbers \(c_1, \ldots, c_n\) and \(C_1, \ldots, C_n\), such that

\[(1)\quad \text{On each partition interval } x_{j-1} \leq x \leq x_j \text{ the inequalities } c_j \leq f(x) \leq C_j \text{ hold, } j = 1, \ldots, n;\]

\[(2)\quad \sum_{j=1}^{n} C_j \cdot (\Delta x_j) - \sum_{j=1}^{n} c_j \cdot (\Delta x_j) < \epsilon, \text{ where } \Delta x_j = x_j - x_{j-1}, \text{ the length of the } j\text{-th interval of the partition.}\]

Here (1) says that the rectangles whose bases are \([x_{j-1}, x_j]\) and whose heights are \(c_j\) and \(C_j\) respectively are inscribed under and circumscribed over the area under the graph of \(f(x)\) on \([x_{j-1}, x_j]\), respectively, while (2) says that the difference between the “area that is too large” and the “area that is too small” will be as small as you please, provided that the partition \(a = x_0 < \cdots < x_n = b\) is chosen correctly. Now those “areas of unions of rectangles” are really the integrals of step functions: if we define two step functions \(s(x)\) and \(S(x)\) by

\[
s(x) = \begin{cases} 
  c_j & \text{for } x_{j-1} \leq x < x_j, \quad j = 1, \ldots, n - 1; \\
  c_n & \text{for } x_{n-1} \leq x \leq x_n 
\end{cases}
\]

\[
S(x) = \begin{cases} 
  C_j & \text{for } x_{j-1} \leq x < x_j, \quad j = 1, \ldots, n - 1; \\
  C_n & \text{for } x_{n-1} \leq x \leq x_n 
\end{cases}
\]

then evidently we have \(s(x) \leq f(x) \leq S(x)\) for all \(a \leq x \leq b\) and

\[
\sum_{j=1}^{n} c_j \cdot (\Delta x_j) = \int_a^b s(x) \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b S(x) \, dx = \sum_{j=1}^{n} C_j \cdot (\Delta x_j).
\]

Since the difference between the two approximating sums is \(\leq \epsilon\), we have the two relations

\[0 \leq \int_a^b [f(x) - s(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b s(x) \, dx \leq \int_a^b S(x) \, dx - \int_a^b s(x) \, dx < \epsilon\]

and these are really the things we need to establish the Riemann-Lebesgue lemma for a Riemann-integrable function \(f(x)\).

We assume that the function \(f(x)\) has been given. This is a formal proof of a limiting relation: we will show that given any permissible error \(\epsilon > 0\) there exists an \(A > 0\), such that if \(|\alpha| \geq A\) then

\[\left| \int_a^b f(x) \cos(\alpha x + \beta) \, dx \right| < \epsilon.\]

First, we apply the discussion of Riemann integrals given above to the function \(f(x)\) and the error \(\frac{\epsilon}{2}\), so that we find a partition \(a = x_0 < x_1 < \cdots < x_n = b\) and a simple function \(s(x)\) such that

\[
s(x) \leq f(x) \quad \text{for all } a \leq x \leq b
\]

\[
0 \leq \int_a^b [f(x) - s(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b s(x) \, dx \leq \int_a^b S(x) \, dx - \int_a^b s(x) \, dx < \epsilon.
\]

\(^{(1)}\) Actually, we’re using the lower Darboux integral, not that anybody would notice.

\(^{(2)}\) There are nice pictures of these approximations in the current calculus textbook, J. Stewart, Calculus: Early Transcendentals, 4th ed., pp. 368–371.
We then have (we’re simply “adding zero”)
\[
\int_a^b f(x) \cos(\alpha x + \beta) \, dx = \int_a^b [f(x) - s(x)] \cos(\alpha x + \beta) \, dx + \int_a^b s(x) \cos(\alpha x + \beta) \, dx
\]
\[
\left| \int_a^b f(x) \cos(\alpha x + \beta) \, dx \right| \leq \left| \int_a^b [f(x) - s(x)] \cos(\alpha x + \beta) \, dx \right| + \left| \int_a^b s(x) \cos(\alpha x + \beta) \, dx \right| .
\]

We shall be finished if we show how to control the size of both terms on the r. h. side of the last set-off line above. The first term is easy: since the absolute value of an integral can be estimated by the integral of the absolute value of the integrand, we have
\[
\left| \int_a^b [f(x) - s(x)] \cos(\alpha x + \beta) \, dx \right| \leq \int_a^b |f(x) - s(x)| \cdot 1 \, dx = \int_a^b |f(x) - s(x)| \, dx < \frac{\epsilon}{2}
\]
(note that we used the fact that \( s(x) \leq f(x) \) implies that \( 0 \leq f(x) - s(x) = |f(x) - s(x)| \)). But the second term is also easy, because it involves the step function \( s(x) \), and we know by §2 above that the Riemann-Lebesgue lemma holds for step functions. Consequently, we know that there exists an \( A \) for which \( |\alpha| \geq A \) implies \( \left| \int_a^b s(x) \cos(\alpha x + \beta) \, dx \right| < \frac{\epsilon}{2} \). Putting the two things together gives
\[
|\alpha| \geq A \implies \left| \int_a^b f(x) \cos(\alpha x + \beta) \, dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
and that establishes the Riemann-Lebesgue lemma for general Riemann-integrable functions.

4. Easy Cases, Extensions and Non-Extensions: It’s easy to see that the Riemann-Lebesgue lemma should be true for differentiable functions, because one can simply integrate by parts:
\[
\int_a^b f(x) \cos(\alpha x + \beta) \, dx = \frac{f(x) \sin(\alpha x + \beta)}{\alpha} \bigg|_a^b - \frac{1}{\alpha} \int_a^b f'(x) \sin(\alpha x + \beta) \, dx .
\]
The two terms can be estimated by \( 2 \cdot \max(|f(a)|, |f(b)|) \cdot \frac{1}{|\alpha|} \) and \( \frac{1}{|\alpha|} \int_a^b |f'(x)| \, dx \) respectively, and both of these estimates tend to zero like \( 1/|\alpha| \). What is surprising about the Riemann-Lebesgue lemma (in the form in which one needs it) is that \( f(x) \) doesn’t even have to be continuous. Of course there is no estimate of the speed at which the integral will tend to 0 as \( |\alpha| \) gets large.

It is easy to extend the result of §3 to improperly-Riemann-integrable functions \( f(x) \), provided that the improper integral of \( |f(x)| \) also exists. For example, if \( \int_a^\infty f(x) \, dx \) exists and \( \int_a^\infty |f(x)| \, dx \) also converges, then given any \( \epsilon > 0 \) we may find \( b > a \) for which \( \int_a^b |f(x)| \, dx < \frac{\epsilon}{2} \) and then write
\[
\int_a^\infty f(x) \cos(\alpha x + \beta) \, dx = \int_a^b f(x) \cos(\alpha x + \beta) \, dx + \int_b^\infty f(x) \cos(\alpha x + \beta) \, dx
\]
\[
\left| \int_a^\infty f(x) \cos(\alpha x + \beta) \, dx \right| \leq \left| \int_a^b f(x) \cos(\alpha x + \beta) \, dx \right| + \left| \int_b^\infty f(x) \cos(\alpha x + \beta) \, dx \right|
\]
\[
\leq \left| \int_a^b f(x) \cos(\alpha x + \beta) \, dx \right| + \int_b^\infty |f(x) \cos(\alpha x + \beta)| \, dx
\]
\[
\leq \left| \int_a^b f(x) \cos(\alpha x + \beta) \, dx \right| + \frac{\epsilon}{2} .
\]

There exists an \( A \) for which \( |\alpha| > A \) will make the first term \( < \frac{\epsilon}{2} \) and thus make
\[
\left| \int_a^\infty f(x) \cos(\alpha x + \beta) \, dx \right| < \epsilon .
\]
A similar argument, *mutatis mutandis*, can be made for functions like \( \frac{1}{\sqrt{x}} \), for which the integral \( \int_0^1 \frac{dx}{\sqrt{x}} \) exists improperly.

One has to be a bit more careful, however, in cases involving improper Riemann integrals that converge because of some sort of cancellation phenomenon; the reason is that the Riemann-Lebesgue lemma is itself true only because of cancellation (for large values of \(|\alpha|\), most of the integral of a step function “cancels itself out” and that’s what makes the size of the integral small). For example, what makes the famous Fresnel integrals of geometrical optics

\[
\int_{-\infty}^{\infty} \cos(x^2) \, dx = \int_{-\infty}^{\infty} \sin(x^2) \, dx = \sqrt{\frac{\pi}{2}}
\]

be convergent integrals is a lot of cancellation (of the area of arches above the \( x \)-axis with that of arches below the \( x \)-axis). Consider the integral

\[
\int_{-\infty}^{\infty} \sin(x^2) \cos(\alpha x) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} [\sin(x^2 + \alpha x) + \sin(x^2 - \alpha x)] \, dx
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} [\sin(x^2 + \alpha x + \alpha^2/4 - \alpha^2/4) + \sin(x^2 - \alpha x + \alpha^2/4 - \alpha^2/4)] \, dx \quad \text{(completing the square)}
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} [\sin((x + \alpha/2)^2/4) \cos(\alpha^2/4) - \cos((x + \alpha/2)^2/4) \sin(\alpha^2/4)

+ \sin((x - \alpha/2)^2/4) \cos(\alpha^2/4) - \cos((x - \alpha/2)^2/4) \sin(\alpha^2/4)] \, dx \quad \text{(addition/subtraction formulas)}
\]

\[
= \frac{1}{2} \left[ \sqrt{\frac{\pi}{2}} \cos(\alpha^2/4) - \sqrt{\frac{\pi}{2}} \sin(\alpha^2/4) + \sqrt{\frac{\pi}{2}} \cos(\alpha^2/4) - \sqrt{\frac{\pi}{2}} \sin(\alpha^2/4) \right] \quad \text{(translate the integrals)}
\]

\[
= \sqrt{\frac{\pi}{2}} \cdot [\cos(\alpha^2/4) - \sin(\alpha^2/4)] = \sqrt{\frac{\pi}{2}} \cdot \{\sqrt{2} \cos(\alpha^2/4 + \pi/4)\} = \sqrt{\pi} \cos \left( \frac{\alpha^2 + \pi}{4} \right) \quad \text{(addition formula)}.
\]

Evidently this function of \( \alpha \) does *not* tend to zero as \(|\alpha| \to \infty\).