THE LAPLACE EQUATION, HOLOMORPHIC FUNCTIONS, AND RELATED MATTERS—PART I

A course in the elementary theory of functions of one complex variable is not a prerequisite for Math 423. However, the connection between the Laplace equation in \( \mathbb{R}^2 \) and complex-variable theory is so close and so convenient to have available that it would be a shame not to consider it. Moreover, some things that are usually discussed in complex-variable terms really pertain to the Laplace equation (and Euclidean geometry), and extend naturally to \( \mathbb{R}^n \) for \( n \geq 3 \).

1. The Real Plane as 1-Dimensional Complex Space. Most of the following material should be known to you since high school. The real plane (and vector space) \( \mathbb{R}^2 \) can be put in 1-1 correspondence with the complex number field \( \mathbb{C} \): the correspondence is \((x, y) \leftrightarrow x + iy\). (In some expositions of the complex numbers, \( \mathbb{C} \) actually is the real plane \( \mathbb{R}^2 \) equipped with the multiplication operation \((a, b) \cdot (c, d) = (ac - bd, ad + bc)\), and the 1-1 correspondence in that case is in fact the identity mapping.) The real part and imaginary part functions are defined on \( \mathbb{C} \) via this identification: \( \text{Re} [x + iy] = x \) and \( \text{Im} [x + iy] = y \).

The conjugate \( \overline{z} \) of a complex number \( z = x + iy \) is defined by \( \overline{z} = x - iy \); from the standpoint of the identification with \( \mathbb{R}^2 \), conjugation is reflection across the \( x \)-axis, and it is a real-linear transformation. If \( \mathbb{R} \) is identified with the subfield of \( \mathbb{C} \) consisting of numbers of the form \( x + 0i \), then the correspondence respects the real vector space structures of \( \mathbb{R}^2 \) and \( \mathbb{C} \) when \( \mathbb{R} \) is thought of as a subfield of \( \mathbb{C} \). One also has \( \text{Re} [z] = \frac{z + \overline{z}}{2} \) and \( \text{Im} [z] = \frac{z - \overline{z}}{2i} \). Moreover, the usual inner-product structure of \( \mathbb{R}^2 \) carries over to \( \mathbb{C} \) in a rather charming way, in that if \( z = x + iy \) and \( w = u + iv \) then

\[
\overline{z}w = (x - iy)(u + iv) = (xu + yv) + i(xy - yu) = (x, y) \cdot (u, v) + i \begin{vmatrix} x & y \\ u & v \end{vmatrix};
\]

both the inner product and the determinant (as functions of pairs of vectors) can be written using complex operations on the corresponding complex numbers. Conjugation preserves multiplication, as the comparison

\[
\frac{(a + bi) \cdot (c + di)}{(a + bi)} = (ac - bd) + i(ad + bc) = (ac - bd) - i(ad + bc)
\]

demonstrates; it follows that also \( \frac{\overline{z}}{w} = \overline{\frac{z}{w}} \).

Everybody knows how to install polar coordinates in the real plane, and in terms of these, \( x + iy = r(\cos \theta + i \sin \theta) \). The Euler power-series computation, familiar to everyone from their o. d. e. courses, shows that \( r(\cos \theta + i \sin \theta) = re^{i\theta} \), and more generally that there is an exponential function defined on \( \mathbb{C} \) taking its values in \( \mathbb{C} \setminus \{0\} \) and defined in terms of elementary real functions by

\[
e^z = e^{x + iy} = e^x \cos y + i e^x \sin y.
\]

The “law of exponents” is valid for \( e^z \); one has \( e^z \cdot e^w = e^{z+w} \) by virtue of the addition formulas for the real exponential and the real trigonometric functions. From the standpoint of polar coordinates, \( re^{i\theta} \cdot re^{i\psi} = (r\rho) \cdot e^{i(\theta + \psi)} \). “In multiplication of complex numbers, lengths multiply and (polar) angles add.” For this reason (as well as others), the reciprocal of \( re^{i\theta} \) is \( \frac{1}{r} e^{-i\theta} \). For other information, see Strauss, pp. 112–113.

There is an absolute-value function for complex numbers, namely \(|x + iy| = \sqrt{x^2 + y^2} \), which is the “\( r \)” of polar coordinates and can be identified with the usual Euclidean norm (“length”) of vectors in \( \mathbb{R}^2 \). It satisfies the triangle inequality because vector lengths do in \( \mathbb{R}^2 \), and evidently \( z = 0 \) if and only if \(|z| = 0 \). Because “lengths multiply and (polar) angles add,” one has \(|zw| = |z| \cdot |w| \) and therefore also \( \left| \frac{z}{w} \right| = \frac{|z|}{|w|} \), except of course when \( w = 0 \).

It is just as easy to think about complex-valued functions of real variables as it is to think about real-valued functions of real variables: for example, a complex-valued function \( f(x, y) = u(x, y) + iv(x, y) \)
can simply be thought of as a pair of real-valued functions \( u(x, y) \) and \( v(x, y) \), or (via the identification of \( \mathbb{R}^2 \) with \( \mathbb{C} \)) as a vector-valued function of \((x, y) \in \mathbb{R}^2 \); a physicist might call it a vector field in the plane. One can multiply complex-valued functions by complex constants, or in fact multiply one complex-valued function by another, and such elementary relations that one learns in calculus courses as the linearity of differentiation and the product and quotient rules carry over to complex-valued functions with no substantial alteration in their statements or proofs. It is possible to state a chain rule, but only by going back to the notion of a \( \mathbb{R}^2 \)-valued function; so we’ll leave that alone for the time being.

2. The operators \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \overline{z}} \). These are sometimes written as \( \partial_\zeta \) or \( \partial \) and \( \partial_\overline{\zeta} \) or \( \overline{\partial} \) respectively; we shall avoid that convention for the most part, but may still read them aloud as “dee” and “dee-bar” respectively. A computation that I didn’t want to take time for in class may help to show why these names are natural. Suppose \( f(x, y) \) is a continuously differentiable function defined in a open subset \( U \subseteq \mathbb{R}^2 \)—alias \( U \subseteq \mathbb{C} \), suppose \((x_0, y_0)\)—alias \( z_0 = x_0 + iy_0 \)—is a point in \( U \), and \((\Delta x, \Delta y)\)—alias \( \Delta z = \Delta x + i \Delta y \) is a “small change vector.” Then

\[
f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + o\left(\sqrt{\Delta x^2 + (\Delta y)^2}\right)
\]

where the term \( o\left(\sqrt{\Delta x^2 + (\Delta y)^2}\right) \) is a “remainder term” that goes to zero so fast that its quotient by \( \| (\Delta x, \Delta y) \| = \sqrt{\Delta x^2 + (\Delta y)^2} \) still goes to zero as \( \| (\Delta x, \Delta y) \| \to 0 \). Now if we simply transfer (1) to complex notation, writing \( z_0 \) for \((x_0, y_0)\), \( \Delta z = \Delta x + i \Delta y \), \( |\Delta z| = \sqrt{\Delta x^2 + (\Delta y)^2} \) and—crucially—\( \overline{\Delta z} = \Delta x - i \Delta y \), so that \( \Delta x = \frac{\Delta z + \overline{\Delta z}}{2} \) and \( \Delta y = \frac{\Delta z - \overline{\Delta z}}{2i} \), then (1) can be rewritten as

\[
f(z_0 + \Delta z) = f(z_0) + \frac{\partial f}{\partial x}(z_0, y_0) \cdot \Delta z + \frac{\partial f}{\partial y}(z_0, y_0) \cdot \overline{\Delta z} + o(|\Delta z|)
\]

and so, if we define the complex-valued homogeneous first-order linear partial differential operators \( \partial \) and \( \overline{\partial} \) by

\[
\begin{align*}
\frac{\partial}{\partial z} &= \frac{1}{2} \left[ \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right] \\
\frac{\partial}{\partial \overline{z}} &= \frac{1}{2} \left[ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right]
\end{align*}
\]

then (1) takes the complex form (of the approximation property of the differential)

\[
f(z_0 + \Delta z) = f(z_0) + \frac{\partial f}{\partial z}(z_0) \Delta z + \frac{\partial f}{\overline{\partial} z}(z_0) \overline{\Delta z} + o(|\Delta z|) \tag{2}
\]

It is important to note some things. First, the operators \( \partial \) and \( \overline{\partial} \) are defined in terms of \( \partial/\partial x \) and \( \partial/\partial y \) and **have no meaning by themselves**, although there are some theorems that make them seem to be because of the genius of the notation. Second, even if \( f \) is a real-valued function, \( \frac{\partial f}{\partial z} \) and \( \frac{\partial f}{\partial \overline{z}} \) will in general be complex-valued functions and must be treated as such. Third, a function \( f(x, y) \), even if complex-valued, **should not be thought of as a function of \( z \) and \( \overline{z} \)** except under very restrictive circumstances. And there are probably other things one should note.

Because \( \partial_x \) and \( \partial_y \) obey the linearity of differentiation and the product and quotient rules, so do \( \partial_z \) and \( \partial_{\overline{z}} \). One can check that these operators obey a version of the chain rule; the only thing we really want to consider is a chain rule for constant multiples of the argument. If \( \alpha + i\beta \) is a complex constant, then \((\alpha + i\beta)(x + iy) = (ax - \beta y) + i(\beta x + \alpha y)\), and so if \( f(x, y) \) is written as \( f(z) \), then \( f((\alpha + i\beta)z) = \)

\[
f((\alpha + i\beta)z) = f((\alpha + i\beta)(x + iy)) = f(ax - \beta y + i(\beta x + \alpha y)) = \frac{\partial f}{\partial z}(z_0) \cdot \Delta z + \frac{\partial f}{\overline{\partial} z}(z_0) \overline{\Delta z} + o(|\Delta z|).
\]
Let \( f(x - y, x + iy) \) and consequently \( \partial_z f((\alpha + i\beta)z) = \alpha \partial_x f + \beta \partial_y f \) and \( \partial_y f((\alpha + i\beta)z) = -\beta \partial_x f + \alpha \partial_y f \).

It follows that
\[
\frac{\partial f((\alpha + i\beta)z)}{\partial z} = \frac{1}{2} \left[ \left( \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial y}{\partial y} \right) - i \left( -\beta \frac{\partial f}{\partial x} + \alpha \frac{\partial f}{\partial y} \right) \right] = (\alpha + i\beta) \frac{\partial f}{\partial z} \mid_{(\alpha + i\beta)z}
\]

and similarly
\[
\frac{\partial f((\alpha + i\beta)z)}{\partial \overline{z}} = \frac{1}{2} \left[ \left( \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial y}{\partial y} \right) + i \left( -\beta \frac{\partial f}{\partial x} + \alpha \frac{\partial f}{\partial y} \right) \right] = (\alpha - i\beta) \frac{\partial f}{\partial \overline{z}} \mid_{(\alpha + i\beta)z}.
\]

In particular, one has a very rudimentary chain rule, and one observes that if \( \frac{\partial f(z)}{\partial z} = 0 \) then also \( \frac{\partial f(cz)}{\partial \overline{z}} = 0 \) for any complex constant \( c \), and similarly for \( \partial_z \).

The power rule for natural-number powers \( f(z)^n \) of a function \( f(z) \) follows by induction on \( n \) from the product rule; we therefore have
\[
\frac{\partial f^n}{\partial z} = n \cdot f(z)^{n-1} \frac{\partial f}{\partial z} \quad \text{and} \quad \frac{\partial f^n}{\partial \overline{z}} = n \cdot f(z)^{n-1} \frac{\partial f}{\partial \overline{z}}
\]

for all \( n \in \mathbb{Z} \). Since it is trivial to verify that \( \frac{\partial z}{\partial z} = 1, \frac{\partial \overline{z}}{\partial z} = 0, \frac{\partial z}{\partial \overline{z}} = 0 \) and \( \frac{\partial \overline{z}}{\partial \overline{z}} = 1 \), we have
\[
\frac{\partial z^n}{\partial z} = 0 \quad \text{and} \quad \frac{\partial z^n}{\partial \overline{z}} = 0
\]

for all integers \( n \). One verifies directly that
\[
\frac{\partial e^z}{\partial z} = \frac{1}{2} \left[ \frac{\partial e^{x+iy}}{\partial x} - i \frac{\partial e^{x+iy}}{\partial y} \right] = \frac{1}{2} \left[ e^x e^{iy} - i e^x e^{iy} \right] = e^x e^{iy} = e^z \quad \text{and}
\]
\[
\frac{\partial e^z}{\partial \overline{z}} = \frac{1}{2} \left[ \frac{\partial e^{x+iy}}{\partial x} + i \frac{\partial e^{x+iy}}{\partial y} \right] = \frac{1}{2} \left[ e^x e^{iy} + i e^x e^{iy} \right] = 0,
\]

and the chain rule then gives \( \frac{\partial e^{cz}}{\partial z} = ce^{cz} \) and \( \frac{\partial e^{cz}}{\partial \overline{z}} = 0 \) for any complex constant \( c \).

In the course of computing the Laplace operator for \( \mathbb{R}^2 \) in polar coordinates, we found that
\[
\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}
\]

and
\[
\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}.
\]

We can therefore write the operators \( \partial_z \) and \( \partial_{\overline{z}} \) in polar-coordinate form as
\[
\frac{\partial f}{\partial z} = \frac{1}{2} \left\{ \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right\} = \frac{1}{2} \left\{ \left[ \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right] - i \left[ \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right] \right\} = \frac{1}{2} \left[ (\cos \theta - i \sin \theta) \frac{\partial f}{\partial r} - i \left( \cos \theta - i \sin \theta \right) \frac{\partial f}{\partial \theta} \right] = \frac{1}{2} \left[ \frac{\partial f}{\partial r} - i \frac{\partial f}{\partial \theta} \right]
\]

and similarly
\[
\frac{\partial f}{\partial \overline{z}} = \frac{1}{2} \left\{ \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right\} = \frac{1}{2} \left\{ \left[ \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right] + i \left[ \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right] \right\} = \frac{1}{2} \left[ (\cos \theta + i \sin \theta) \frac{\partial f}{\partial r} + i \left( \cos \theta + i \sin \theta \right) \frac{\partial f}{\partial \theta} \right] = \frac{1}{2} \left[ \frac{\partial f}{\partial r} + i \frac{\partial f}{\partial \theta} \right].
\]
As we shall see, these make it very easy to transform the Laplace operator in \( \mathbb{R}^2 \) into polar coordinates. For now, we shall be content to make a couple of observations. The first is that if we write a real but perhaps noninteger power of \( z = re^{i\theta} \) by writing \( z = re^{i\theta} \) and then formally writing \( z^\alpha = r^\alpha e^{i\alpha \theta} \) (which is \( r^\alpha (\cos \alpha \theta + i \sin \alpha \theta) \)), then

\[
\frac{\partial z^\alpha}{\partial z} = \frac{1}{2z} \left[ \frac{\partial f}{\partial r} - i \frac{\partial f}{\partial \theta} \right] = \frac{1}{2z} \left[ r \cdot \alpha r^{\alpha-1} e^{i\alpha \theta} - i \cdot (r^\alpha \cdot i e^{i\alpha \theta}) \right] = \frac{1}{2re^{i\theta}} \left[ \alpha r^\alpha e^{i\alpha \theta} + r^\alpha \cdot \alpha e^{i\alpha \theta} \right] = \alpha z^{\alpha-1}
\]

while

\[
\frac{\partial z^\alpha}{\partial \overline{z}} = \frac{1}{2z} \left[ r \frac{\partial f}{\partial r} + i \frac{\partial f}{\partial \theta} \right] = \frac{1}{2z} \left[ r \cdot \alpha r^{\alpha-1} e^{i\alpha \theta} + i \cdot (r^\alpha \cdot \alpha e^{i\alpha \theta}) \right] = \frac{1}{2re^{-i\theta}} \left[ \alpha r^\alpha e^{i\alpha \theta} - r^\alpha \cdot \alpha e^{i\alpha \theta} \right] = 0,
\]

extending what we knew about powers of \( z \) to fractional powers. The second is that we can make extensions of the real logarithm function to complex values of the argument by observing that if \( z = re^{i\theta} \), then—formally at least—it must be that \( \log z = \log r + i\theta \), where \( \theta \) is the polar angle. Since with this definition we have for \( f(z) = \log r + i\theta \)

\[
\frac{\partial f}{\partial z} = \frac{1}{2z} \left[ r \frac{\partial f}{\partial r} - i \frac{\partial f}{\partial \theta} \right] = \frac{1}{2z} \left[ r \left( \frac{1}{r} + i0 \right) - i \cdot (0 + i) \right] = \frac{1}{2z} \cdot 2 = \frac{1}{z}
\]

while

\[
\frac{\partial \log z}{\partial \overline{z}} = \frac{1}{2z} \left[ r \frac{\partial f}{\partial r} + i \frac{\partial f}{\partial \theta} \right] = \frac{1}{2z} \left[ r \cdot \left( \frac{1}{r} + i0 \right) + i \cdot (0 + i) \right] = \frac{1}{2z} (1 - 1) = 0,
\]

we once again have a “function of \( z \) only.” However, it should be noted that the fractional powers of \( z \) and this logarithm are not defined intrinsically on \( \mathbb{C} \) (or \( \mathbb{R}^2 \)), as the integer powers of \( z \) are: since the polar coordinate \( \theta \) is not uniquely defined, a choice has to be made, and in a differentiable way. The usual approach is to work on a slit plane obtained by removing a ray issuing from 0. For example, one can slit the plane along the negative real = \( x \)-axis, defining the polar angle to lie in \( -\pi < \theta < \pi \), or along the negative imaginary = \( y \)-axis, defining the polar angle to lie in \( -\pi/2 < \theta < 3\pi/2 \). Other choices are of course possible; an advantage of the latter choice is that \( \theta = \text{Arctan} (y/x) \) for \( x > 0 \) and \( \theta = \frac{\pi}{2} - \text{Arctan} (x/y) \) for \( y > 0 \). That relation was (implicitly) used in constructing the Poisson kernel for the upper half-plane earlier in this course.

3. If you already know complex variables . . . . If \( f = u + iv \) with \( u \) and \( v \) real, then the relation \( \frac{\partial f}{\partial \overline{z}} = 0 \) is equivalent to the two Cauchy-Riemann equations:

\[
\frac{\partial (u + iv)}{\partial \overline{z}} = \frac{1}{2} \left[ \frac{\partial (u + iv)}{\partial x} + i \frac{\partial (u + iv)}{\partial y} \right] = \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]
\]

with the real and imaginary parts separated on the r. h. side, and so if this expression equals zero then separately

\[
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0
\]

and these are the Cauchy-Riemann equations. On the other hand, if this happens then we may divide (2) by \( \Delta z \), with no \( \Delta \overline{z} \) term present, and get

\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \left[ \frac{\partial f}{\partial z} \bigg|_{z_0} + o(|\Delta z|) \right] = \frac{\partial f}{\partial z} \bigg|_{z_0};
\]

we have \( \frac{\partial f}{\partial z} = f'(z_0) \) by definition of the derivative for complex-valued functions. Thus \( \frac{\partial f}{\partial z} \) “really is the derivative with respect to \( z \),” if the derivative exists. (With a little work, one can derive the necessity of
the Cauchy-Riemann equations from (2) by a similar calculation.) Thus \( \frac{\partial f}{\partial \overline{z}} \equiv 0 \) characterizes holomorphic functions among \( \mathcal{C}^1 \)-functions on open \( U \subseteq \mathbb{C} \). If you believe, e.g., that \( \frac{\partial f}{\partial y} \equiv 0 \) characterizes the functions that depend on \( x \) but not on \( y \), then it seems natural to say that “the holomorphic functions are those that depend only on \( z \) and not on \( \overline{z} \).”

4. **Factoring the Laplace Operator.** Life with the 1-dimensional wave equation was simplified enormously by the factorization(s)

\[
\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \left( \frac{\partial}{\partial t} \pm c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} \pm c \frac{\partial}{\partial x} \right),
\]

from which the D’Alembert solution of the wave equation flows in the manner Strauss describes on pp. 32–33. To do the same thing with the Laplace operator we would need a square root of minus one, but of course we cannot exactly mimic the D’Alembert solution of the wave equation like \( f(x + iy) + g(x - iy) \) directly. On the other hand, \( f(x + iy) + g(x - iy) \) looks like \( f(z) + g(\overline{z}) \), and if we regard “\( f(z) \)” as meaning a function that “does not depend on \( \overline{z} \),” i.e., satisfies \( \frac{\partial f}{\partial \overline{z}} = 0 \), then at least we see that any such function does satisfy the Laplace equation. Similarly, because the coefficients of the Laplace operator are real, we see that if \( f = u + iv \) satisfies \( \frac{\partial f}{\partial \overline{z}} = 0 \), which implies that \( \nabla^2 f = \nabla^2 u + i\nabla^2 v = 0 \), then the real and imaginary parts satisfy the Laplace equation separately. This observation gets us the basic building blocks for solutions of the Laplace equation quite painlessly when we observe that since \( \frac{\partial \alpha}{\partial \overline{z}} = 0 \) for every real \( \alpha \), the functions \( r^\alpha \cos \alpha \theta \) and \( r^\alpha \sin \alpha \theta \) satisfy the Laplace equation. In particular, for \( n \in \mathbb{Z} \) the functions \( r^n \cos n\theta \) and \( r^n \sin n\theta \), which are well-defined on \( \mathbb{R}^2 \) without making a choice of the polar angle since they are respectively \( \text{Re} \{z^n\} \) and \( \text{Im} \{z^n\} \), will be of fundamental importance. (In rectangular coordinates these are such “harmonic polynomials” as \( \text{Re} \{z^4\} = \text{Re} \{x^4 + 4ix^3 y - 6x^2 y^2 - 4ixy^3 + y^4\} = x^4 - 6x^2 y^2 + y^4 \) and \( \text{Im} \{z^4\} = 4x^3 y - 4xy^3 \), which might seem to require a much less systematic construction. Certainly the construction of their analogues in dimensions higher than 2 presents some interesting problems.) Similarly, the real part \( \log r = \frac{1}{2} \log(x^2 + y^2) \) of the function \( \log z = \log r + i\theta \), which is defined independently of the choice of \( \theta \), satisfies the Laplace equation in \( \mathbb{C} \setminus \{0\} \) and will have a fundamental rôle to play. (The imaginary part \( \theta \) also satisfies the Laplace equation, but we have to be careful of the choice of “branch” of this “multiple-valued function.” Nonetheless, we have implicitly used it in the construction of the Poisson kernel for the upper half-plane.)

An immediate advantage of factorizing the Laplace operator is an easier route to its polar-coördinate form:

\[
\nabla^2 u = 4 \frac{\partial}{\partial z} \left[ \frac{\partial u}{\partial z} \right] = 4 \frac{\partial}{\partial z} \left[ \frac{1}{2z} \left( r \frac{\partial u}{\partial r} - i \frac{\partial u}{\partial \theta} \right) \right] = 4 \left\{ \frac{\partial}{\partial z} \left[ \frac{1}{2z} \right] \cdot i \left( \frac{\partial u}{\partial r} - i \frac{\partial u}{\partial \theta} \right) + \frac{1}{2z} \frac{\partial}{\partial z} \left[ r \frac{\partial u}{\partial r} - i \frac{\partial u}{\partial \theta} \right] \right\} \quad (\partial_r \text{ obeys the product rule})
\]

\[
= 4 \left\{ \frac{1}{2z} \frac{\partial}{\partial z} \left[ r \frac{\partial u}{\partial r} - i \frac{\partial u}{\partial \theta} \right] \right\} \quad (\text{since } \frac{\partial}{\partial z} \left[ \frac{1}{2z} \right] = 0)
\]

\[
= 2 \left\{ \frac{1}{z} \frac{\partial}{\partial z} \left[ \frac{r}{2z} \frac{\partial u}{\partial r} + i \frac{\partial u}{\partial \theta} \right] \right\} \quad (\text{since } \frac{\partial}{\partial z} \left[ \frac{1}{z} \right] = 0)
\]

\[
= \frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \quad (\text{since } z\overline{z} = |z|^2 = r^2).
\]

This agrees with the results of real-variable chain-rule calculations, e.g., Strauss’s (5), p. 151.
5. The Poisson Formula in $\mathbb{R}^2$ and its Consequences. The derivation of the Poisson formula is in Strauss’s §6.3, where you may read all about it. I have only a few things to add.

1. The function $P(r, t) = \frac{1-r^2}{1-2r \cos t + r^2}$ is usually called the Poisson kernel for the disc. The apparent choice of the disc of radius 1 is only a normalization, since

$$\frac{a^2-r^2}{a^2-2ar \cos t + r^2} = \frac{1-(\frac{r}{a})^2}{1-2 \left(\frac{r}{a}\right) \cos t + \left(\frac{r}{a}\right)^2}.$$ 

Thus, unless one absolutely has to look at the individual functions in the Poisson kernel carefully, one can abbreviate Strauss’s formula (13), p. 161 to

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) P(r/a, \theta - \phi) \, d\phi.$$ 

Moreover, since one can install polar coordinates with their “pole” at any point of the plane, the Poisson kernel can be used to solve the Dirichlet boundary-value problem for $\nabla^2 u = 0$ for any disc in the plane.

2. From the formula for the Poisson kernel (but perhaps not from the series that we summed to construct it), it is obvious—look at the largest and smallest values of the denominator, which are perfect squares of binomials—that it satisfies the following inequalities, called the Harnack estimates:

$$0 < \frac{a-r}{a+r} \leq P(r/a, t) \leq \frac{a^2-r^2}{a^2-2ar \cos t + r^2} \leq \frac{a+r}{a-r} \text{ for } 0 \leq r < a.$$ 

In particular, it is positive for all (real) values of $t$. Those inequalities are sharp, with the under-estimate being attained for $t = \pi$ and the over-estimate being attained for $t = 0$. In particular, it blows up like $\frac{1}{a-r}$ as $r \to a^-$ for $t = 0$. However, if $|t|$ is kept bounded away from zero—say $|t| > \delta > 0$—then using the well-known (and sharp) inequality $2ar \leq a^2 + r^2$ of the arithmetic and geometric means, we can make the (sharp) estimate

$$a^2-2ar \cos t + r^2 \geq a^2-2ar \cos \delta + r^2 \geq a^2-(a^2+r^2) \cos \delta + r^2 = (a^2+r^2)(1-\cos \delta) = (a^2+r^2)2 \sin^2 \left(\frac{\delta}{2}\right).$$ 

Thus for $|t| > \delta$ we have $P(r/a, t) \leq \frac{a^2-r^2}{2(a^2+r^2) \sin^2(\delta/2)}$, which tends to 0 independently of $t$ as $r \to a^-$. A graph of the Poisson kernel as a function of the “angle” $t$ therefore looks very much like the graph of the source function for the diffusion equation as depicted in Strauss’s Figure 1, p. 49. Of course it is bounded away from zero, and the peak at $t = 0$ is repeated every $2\pi$ units—an irrelevant fact if one is only integrating around a circle. But the Poisson integral

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2-r^2)}{a^2-2ar \cos(\theta - \phi) + r^2} h(\phi) \, d\phi$$

does average the values of $h(\phi)$ around the angle $\phi = \theta$, just as the source function $S(x-y)$ averages the values of the initial-value function $\varphi(y)$ in the discussion on Strauss’s p. 49. Because of its positivity, it does a much better job of averaging than does the Dirichlet kernel, which depends on cancellation to make Fourier series converge to the functions that generated their coefficients. (The Dirichlet kernel is pictured in Strauss’s Figure 1 on p. 133.) It is thus fairly easy to show that if $h(\theta)$ is a continuous function on the circle of radius $a$, then the function on the closed disc $r \leq a$ defined by

$$u(r, \theta) = \begin{cases} h(\theta) & \text{if } r = a \\ \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2-r^2)}{a^2-2ar \cos(\theta - \phi) + r^2} h(\phi) \, d\phi & \text{if } 0 \leq r < a \end{cases}$$

is continuous at every point of the closed disc including the boundary points, i.e., that no matter how $(r, \theta)$ approaches an arbitrary point $(r_0, \theta_0)$ in the disc (even if $r_0 = a$), one will have $u(r, \theta) \to u(r_0, \theta_0)$. 


(3) On the other hand, from the series that we summed to get the Poisson kernel (but perhaps not from
the closed-form formula), it is obvious that
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2)}{a^2 - 2ar \cos(\theta - \phi) + r^2} d\phi = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \cos(n(\theta - \phi))}{a^2 - 2ar \cos(\theta - \phi) + r^2} \right\} d\phi = 1.
\]
Thus the integral of the Poisson kernel is independent of \( r < a \) and is equal to 1 (at least in the presence
of the normalizing factor \( \frac{1}{2\pi} \) ) for all values of \( r < a \). So at any point of a disc in \( \mathbb{R}^2 \), the value of the solution
\[
u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2)}{a^2 - 2ar \cos(\theta - \phi) + r^2} h(\phi) d\phi
\]
of the Dirichlet boundary-value problem appears as a weighted average of the boundary values. In
particular, at the center of the disc, \( u(0) = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi \) (note that the “zero” here is the center of
the disc, which may not be the origin of \( xy \)-coordinates) appears simply as the mean of the values on
the boundary. As we shall see, this property actually characterizes solutions of the Laplace equation
\( \nabla^2 u = 0 \) among continuous functions: this condition on integrals has the surprising property of turning
mere continuity into high differentiability.

(4) Solutions of the Laplace equation are characterized by “being unaltered by the Poisson kernel.” Specifically:
suppose \( U \subseteq \mathbb{R}^2 \) is an open set, and suppose that \( u(x, y) \) is a solution of \( \nabla^2 u = 0 \) defined in
\( U \). Let \( D \) be any closed disc (one that includes its boundary) that is entirely contained in \( U \); suppose
\( D \) has center \((x_0, y_0)\) and radius \( a > 0 \). Install polar coordinates with their pole at \((x_0, y_0)\),
so that points in the plane can be located via the equations \( x = x_0 + r \cos \theta, y = y_0 + r \sin \theta \). Let
\( h(\phi) = u(x_0 + a \cos \phi, y_0 + a \sin \phi) \), the values of \( u \) on the boundary of \( D \), and solve the Dirichlet
problem on \( D \) using the Poisson kernel with these boundary values by setting
\[
v(r, \theta) = \begin{cases} h(\theta) = u(x_0 + a \cos \phi, y_0 + a \sin \phi) & \text{if } r = a \\ \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2)}{a^2 - 2ar \cos(\theta - \phi) + r^2} h(\phi) d\phi & \text{if } 0 \leq r < a. \end{cases}
\]
The new function \( v \) is continuous on the closed disc, satisfies the Laplace equation inside \( D \) and takes
the same values on the circle bounding \( D \) as \( u \) did, which implies that the functions \( \pm(u - v) \) are
continuous on the closed disc, satisfy the Laplace equation inside \( D \) and take identically-zero boundary
values. By even the weak maximum principle, \( \pm(u - v) \leq 0 \) inside \( D \), which can happen if and only if
\( u(x, y) = v(x, y) \) at all points inside \( D \). Thus the Poisson kernel “gives back \( u \)” in all discs contained
(together with their boundaries) in \( U \). On the other hand, suppose \( u(x, y) \) is only known to be a
continuous function in \( U \), but that it is known to have the property that the equation
\[
u(x_0 + r \cos \theta, y_0 + r \sin \theta) \equiv \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) d\phi
\]
holds identically in every closed disc with center \((x_0, y_0)\) and radius \( a > 0 \) that is entirely contained in
\( U \). Looking at the r. h. side of this equation, we see that \( u \) is a harmonic function; but there is some such
disc centered on every point of \( U \); because \( U \) is open; so \( u(x, y) \) is just as differentiable as the Poisson
kernel is—that is, infinitely differentiable—and satisfies \( \nabla^2 u = 0 \) throughout \( U \). We have proved

**Theorem:** Let \( u(x, y) \) be a continuous function defined on an open set \( U \subseteq \mathbb{R}^2 \). In order that \( u \) be \( C^\infty \)
and satisfy the Laplace equation \( \nabla^2 u = 0 \), it is necessary and sufficient that it have the property that the equation
\[
u(x_0 + r \cos \theta, y_0 + r \sin \theta) \equiv \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) d\phi
\]
hold identically (in polar coordinates with pole at \((x_0, y_0)\)) in every closed disc with center \((x_0, y_0)\)
and radius \( a > 0 \) that is entirely contained in \( U \). Indeed, it is **sufficient** that every point in \( U \) be contained in
the interior of some disc for which \( u(x, y) \) is identically equal to its harmonic extension from its values
on the boundary of that disc.
6. A Weakening of “Subharmonic,” and a Strong Maximum Principle. Let us return to the mean-value property we met in (3) of §5 above, namely that if \( u(x, y) \) is a harmonic function defined on an open set \( U \subseteq \mathbb{R}^2 \), then for every point \( (x_0, y_0) \in U \) and every closed disc of radius \( a > 0 \) lying entirely in \( U \), the mean-value relation

\[
u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos \phi, y_0 + r \sin \phi) \, d\phi\]

holds. Let us make the definition that a continuous real-valued function \( v(x, y) \) defined on an open set \( U \subseteq \mathbb{R}^2 \) will be called submedian in \( U \) if at every point \( (x_0, y_0) \in U \) there exist arbitrarily small radii \( a > 0 \) for which the relation

\[
v(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + r \cos \phi, y_0 + r \sin \phi) \, d\phi\]

holds: that is, at each point there are circles of arbitrarily small radius (not necessarily all small circles) centered on the point, such that the average value of \( v \) over the circle is larger than or equal to the value of \( v \) at the center. Evidently if \( u(x, y) \) is a harmonic function in \( U \), then both \( u \) and \( -u \) are submedian. The interesting property that submedian functions have is

**Proposition ([Strong] Maximum Principle):** Let \( U \subseteq \mathbb{R}^2 \) be a bounded open set, and let \( v(x, y) \) be a continuous function on \( \overline{U} \) (the closure of \( U \), which is the set \( U \) together with its boundary \( \partial U \)) that is submedian in \( U \). Then \( \max\{v(x, y) : (x, y) \in U\} = \max\{v(x, y) : (x, y) \in \partial U\} \); submedian functions attain their maxima on the boundary. Moreover, if \( U \) is connected (in the sense that every two points of \( U \) can be joined by a continuous curve lying entirely in \( U \)) then any submedian function that attains its maximum at a (n interior) point of \( U \) is a constant (equal to that maximum value).

**Proof.** \(^{(1)}\) Suppose first that \( M = \max\{v(x, y) : (x, y) \in U\} > \max\{v(x, y) : (x, y) \in \partial U\} \), so that \( v(x, y) \) attains some value in \( U \) that is larger than any value it attains on the boundary: this assumption leads to an easy contradiction. Let \( S \subseteq U \) be the set on which \( v(x, y) \) attains its maximum: \( S = \{ (x, y) \in U : v(x, y) = M \} \). Let \( (x_0, y_0) \) be a point of \( M \) whose distance to the complement \( \mathbb{R}^2 \setminus U \) of \( U \) is smallest: using the fact that \( S \) is a closed and bounded set and \( \mathbb{R}^2 \setminus U \) is closed, it can be shown that such a point exists. Then if \( a > 0 \) is a radius for which the relation \( M = v(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + r \cos \phi, y_0 + r \sin \phi) \, d\phi \) holds, it is easy to see, using the \( \epsilon-\delta \) definition of continuity, that there is an interval on the circle \( \|(x, y) - (x_0, y_0)\| = a \) on which \( v(x, y) < v(x_0, y_0) = M \), and of course \( v(x, y) \leq M \) everywhere on that circle. But then the value of the integral is \( < M \), so we have the contradiction

\[
M = v(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + r \cos \phi, y_0 + r \sin \phi) \, d\phi < M
\]

following from the assumption that \( M = \max\{v(x, y) : (x, y) \in \overline{U}\} > \max\{v(x, y) : (x, y) \in \partial U\} \), and therefore \( \max\{v(x, y) : (x, y) \in U\} \leq \max\{v(x, y) : (x, y) \in \partial U\} \) must be true.

Now consider the case in which \( U \) is connected, and suppose that \( v(x, y) \) assumes the value \( M = \max\{v(x, y) : (x, y) \in U\} \) at some point(s) of \( U \). Let \( S = \{ (x, y) \in U : v(x, y) = M \} \) as before. If \( S = U \), then \( v(x, y) \equiv M \) is a constant and there is nothing to prove. Otherwise, there are points \( (x_1, y_1) \in U \setminus S \), that is, points in \( U \) that do not belong to \( S \). Connect a point of \( M \) to a point \( (x_1, y_1) \in U \setminus S \) by a curve, i.e., a continuous function \( (x(t), y(t)) \) defined on a real interval \([\alpha, \beta]\), taking its values in \( U \), such that \((x(\alpha), y(\alpha)) \) belongs to \( S \) but \((x(\beta), y(\beta)) \equiv (x_1, y_1) \) does not, and let \((x_0, y_0) = (x(t_0), y(t_0)) \in S \) be the point of \( S \) for which the value of the parameter \( t \) \([\alpha, \beta]\) is largest.\(^{(2)}\) Let \( a > 0 \) be a radius smaller than the distance \( \|(x_0, y_0) - (x_1, y_1)\| \), such that the inequality \( M = v(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi \) holds. There is some point of the curve \((x(t), y(t)) \) with \( t_0 < t < \beta \) for which \((x(t), y(t)) \) is on the circle:

\[(1)\] Some of the topological arguments employed in the proof, like the existence of points of closed-and-bounded sets at which certain minima and maxima are attained, may have to be taken on faith by people who have not had an advanced-calculus course.

\[(2)\] It is intuitively plausible that such a \( t_0 \) exists, and its existence can be demonstrated by a continuity-and-compactness argument.
this happens by the “betweenness theorem” for continuous functions on an interval, because the continuous function \( \|(x(t), y(t)) - (x_1, y_1)\| \) takes a value bigger than \( a \) for \( t = t_0 \), takes the value zero for \( t = \beta \), and must therefore take the value \( a \) somewhere in between. At that point, because \( t > t_0 \), the value of \( v(x, y) \) must be strictly \( < M \). By continuity, there is an interval on the circle about that point for which \( v(x, y) < M \), i.e., an interval \( \gamma < \phi < \delta \) of angles for which \( v(x_0 + a \cos \phi, y_0 + a \sin \phi) < M \), holds: but then again the average 
\[
\frac{1}{2\pi} \int_0^{2\pi} v(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi < M, \text{ and we reach a contradiction as before. That proves the strong maximum property.}
\]

**Corollary:** The strong maximum and minimum properties hold for harmonic functions.

**Proof.** The Poisson kernel shows that if \( u(x, y) \) is harmonic, then both \( u \) and \( -u \) satisfy the submedian condition, and in fact satisfy it for all discs of arbitrary radius contained in their domains.

**Corollary:** Let \( U \subseteq \mathbb{R}^2 \) be an open set. In order that a continuous real-valued function \( v(x, y) \) be submedian, it is necessary and sufficient for every open set \( V \) whose closure \( \overline{V} = V \cup \partial V \subseteq U \) and every continuous real-valued function \( u(x, y) \) defined on \( V \) which is harmonic in \( V \) and such that \( v(x, y) \leq u(x, y) \) at every point \( (x, y) \in \partial V \), the inequality \( v(x, y) \leq u(x, y) \) holds throughout \( V \). This requirement for all \( V \) can be replaced by the corresponding requirement for all discs \( V \) with \( V \cup \partial V \subseteq U \).

**Proof.** Consider a set \( V \) and functions \( u \) and \( v \) as described in the statement. Suppose \( v \) satisfies the definition of submedian. Because the relation \( u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi < M \) holds for all discs in \( V \), the function \( v - u \) satisfies the submedian condition in \( V \) for the same discs for which \( v \) satisfied it. Therefore \( v - u \) takes its maximum on \( \partial V \). But \( v - u \leq 0 \) on \( \partial V \), so \( v - u \leq 0 \) throughout \( V \), which says \( v \leq u \) throughout \( V \). On the other hand, if \( v \) has the property that the inequality 
\[
v(x_0 + r \cos \theta, y_0 + r \sin \theta) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) \, d\phi
\]
is known to hold only for certain discs of arbitrary small radius \( a > 0 \) centered at each point \( (x_0, y_0) \in U \), then by setting \( r = 0 \) we find that \( v \) satisfies the submedian condition
\[
v(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi,
\]
so the condition of “being dominated by every harmonic function that dominates it on the boundary,” even if we require it only for some discs centered at each point of \( U \)—but whose radius can be arbitrarily small—is a stronger requirement than is the submedian condition. Replacing the “all \( V \)” condition by the “all discs \( V \)” condition is handled by the same arguments, the details of which can safely be left to the reader.

**Corollary:** The sum of two submedian functions, or more generally any linear combination \( \alpha_1 v_1 + \alpha_2 v_2 \) where \( \alpha_1, \alpha_2 \geq 0 \) and \( v_1, v_2 \) are submedian, is again a submedian function.

**Proof.** It is obvious that if \( v \) satisfies a submedian condition, then so does \( \alpha v \) for \( \alpha \geq 0 \). It is not obvious that \( v_1 + v_2 \) satisfies a submedian condition if each of \( v_1 \) and \( v_2 \) does, because the radii “\( a \)” of the discs on which \( v_1 \) satisfies the condition of being no larger at the center than its mean over the boundary may be different from the radii “\( a \)” of the corresponding discs for \( v_2 \). However, the corollary proved immediately above showed that if the submedian condition holds as originally stated, then in fact the inequalities 
\[
v_1(x_0 + r \cos \theta, y_0 + r \sin \theta) \leq \frac{1}{2\pi} \int_0^{2\pi} v_1(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) \, d\phi
\]
\[
v_2(x_0 + r \cos \theta, y_0 + r \sin \theta) \leq \frac{1}{2\pi} \int_0^{2\pi} v_2(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) \, d\phi
\]
hold for every closed disc of radius \( a \) and center \( (x_0, y_0) \) contained in \( U \), and adding these inequalities gives
\[ [v_1 + v_2](x_0 + r \cos \theta, y_0 + r \sin \theta) = v_1(x_0 + r \cos \theta, y_0 + r \sin \theta) + v_2(x_0 + r \cos \theta, y_0 + r \sin \theta) \]
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} [v_1(x_0 + a \cos \phi, y_0 + a \sin \phi) + v_2(x_0 + a \cos \phi, y_0 + a \sin \phi)] P(r/a, \theta - \phi) \, d\phi \]

for all such discs, and that condition also characterizes submedian functions.

**Remark:** The same statement can be made for the sums of convergent series \( v = \sum_{k=1}^{\infty} v_k \) whose terms are submedian functions. The notion of “convergent” that one would like to use in such a statement would be “pointwise convergent,” but then it would not be clear what meaning should be attached to the integral of the sum function \( v \). This is another instance in which the availability of Lebesgue integration would make it possible to have more satisfying results. If we had the Lebesgue integral, we could also weaken the condition imposed above that submedian functions be continuous to the condition that they be “upper semicontinuous,” or “could jump down but couldn’t jump up,” allow them to take the value \(-\infty\) (so that, e.g., \( \log r \) would be a submedian function), and have a very interesting and useful collection of results indeed.

**Corollary:** If \( v_1 \) and \( v_2 \) are submedian functions defined on an open set \( V \subseteq \mathbb{R}^2 \), then so is their “pointwise supremum” defined by \( w(x, y) = \max\{v_1(x, y), v_2(x, y)\} \).

**Proof.** One knows that the inequalities
\[ v_1(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v_1(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi \leq \frac{1}{2\pi} \int_0^{2\pi} w(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi \]
\[ v_2(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v_2(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi \leq \frac{1}{2\pi} \int_0^{2\pi} w(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi \]
hold—the second inequality on each line is obvious, since \( v_1 \leq w \) and \( v_2 \leq w \)—for all centers \((x_0, y_0) \in V\) and all \( a > 0 \) for which the closed disc of radius \( a \) is contained in \( V \). But the larger of the two numbers on the l. h. sides of those inequalities is just \( w(x_0, y_0) \), so the two inequalities together imply
\[ w(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} w(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi \]
for all such \((x_0, y_0)\) and \( a > 0 \), and that shows that \( w(x, y) \) is a submedian function.

Note that even if \( v_1 \) and \( v_2 \) are highly differentiable functions, their pointwise supremum—which we shall hereinafter denote by \( v_1 \lor v_2 \), since the symbol “\( \lor \)” reminds one of the graph of the pointwise supremum of two functions whose graphs are lines—may fail to be differentiable. For example, the function \( u(x, y) = y \) and its negative are infinitely differentiable as well as harmonic and therefore submedian, but \( u \lor (-u) = \max\{y, -y\} = |y| \) has a rather bad “crinkle” along the \( x \)-axis.

The notion of submedian function was introduced for two reasons: to show how useful inequalities involving integral means are, and to show that it is possible to talk about subharmonic functions that are not necessarily differentiable. But the definition is really a generalization of the notion of subharmonic function for \( \mathcal{C}^2 \)-functions, which was the requirement that \( \nabla^2 u \geq 0 \) (recall that that condition implies the maximum principle, as we have shown in notes and as Strauss shows on pp. 148–149). So we should finish with the generalities about these functions by showing that if we have a twice-continuously-differentiable function then we have defined nothing new.

**Proposition:** Let \( v(x, y) \) be a twice-continuously-differentiable function defined on an open set \( U \subseteq \mathbb{R}^2 \). Then \( v \) is a submedian function if and only if \( \nabla^2 v \geq 0 \).

**Proof.** Suppose that \( \nabla^2 v \geq 0 \). Let \( D \) be a disc with center \((x_0, y_0)\) and radius \( a > 0 \) which is contained, together with its boundary, in \( U \). Install polar coordinates with center at \((x_0, y_0)\) as usual, and using the Poisson kernel solve the Dirichlet boundary-value problem on \( D \) for the boundary values given by
restricting $v$ to $\partial D$. Let $u$ denote the harmonic function thus obtained, and consider the function $v - u$. Since $\nabla^2[v - u] = \nabla^2 v - \nabla^2 u = \nabla^2 v - 0 = \nabla^2 v \geq 0$, the function $v - u$ obeys the maximum principle and thus takes its maximum on $\partial D$—where its value is zero. So we have $v - u \leq 0$, or $v \leq u$, throughout $D$. Evaluating both sides of that inequality at the center of $D$ we have

$$v(x_0, y_0) \leq u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi$$

which is the submedian condition—and the disc $D$ with $\overline{D} \subseteq U$ was arbitrary.

For the converse, we have to recall how the Taylor series—or second-degree Taylor polynomial—looks for functions of two variables. If $f(x, y)$ is a function defined in an open set $U$ and twice-continuously-differentiable there, and if $P_0 = (x_0, y_0) \in U$, then it is possible to show that for points $(x, y)$ near $P_0$, one has for $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$ the relation$(3)$

$$f(x, y) = f(P_0) + \left\{ \frac{\partial f(P_0)}{\partial x} \Delta x + \frac{\partial f(P_0)}{\partial y} \Delta y \right\}$$

$$+ \frac{1}{2!} \left\{ \frac{\partial^2 f(P_0)}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 f(P_0)}{\partial x \partial y} (\Delta x)(\Delta y) + \frac{\partial^2 f(P_0)}{\partial y^2} (\Delta y)^2 \right\}$$

$$+ R(\Delta x, \Delta y)$$

where the “remainder term” $R(\Delta x, \Delta y)$ is “of order higher than 2” or is “$o(\|P - P_0\|^2)$,” meaning that by making $\|P - P_0\|$ small enough, we can make the quotient $\left| \frac{R(\Delta x, \Delta y)}{\|P - P_0\|^2} \right|$ as small as we please. So in the present context, let $(x_0, y_0) \in U$ be any point, and expand the submedian function $v(x, y)$ in this Taylor series centered at that point, taking the “movable point” $(x, y) = (x_0 + a \cos \theta, y_0 + a \sin \theta)$, so $\Delta x = a \cos \theta$ and $\Delta y = a \sin \theta$. One gets

$$v(x_0 + a \cos \theta, y_0 + a \sin \theta) = v(P_0) + \left\{ \frac{\partial v(P_0)}{\partial x} a \cos \theta + \frac{\partial v(P_0)}{\partial y} a \sin \theta \right\}$$

$$+ \frac{1}{2!} \left\{ \frac{\partial^2 v(P_0)}{\partial x^2} (a \cos \theta)^2 + \frac{\partial^2 v(P_0)}{\partial x \partial y} (a \cos \theta)(a \sin \theta) + \frac{\partial^2 v(P_0)}{\partial y^2} (a \sin \theta)^2 \right\}$$

$$+ R(a \cos \theta, a \sin \theta)$$

and one can now compute averages over the circle of radius $a$ centered at $(x_0, y_0)$ term by term. On the l. h. side, of course, one simply gets the circular mean

$$\frac{1}{2\pi} \int_0^{2\pi} v(x_0 + a \cos \theta, y_0 + a \sin \theta) \, d\theta .$$

On the r. h. side, it is obvious that the circular mean of the constant $v(x_0, y_0)$ is just $v(x_0, y_0)$. The circular mean of the first-order terms is zero:

$$\frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{\partial v(P_0)}{\partial x} a \cos \theta + \frac{\partial v(P_0)}{\partial y} a \sin \theta \right\} \, d\theta = \frac{\partial v(P_0)}{\partial x} \cdot 0 + \frac{\partial v(P_0)}{\partial y} \cdot 0 = 0$$

$(3)$ To my horror, the current RU–NB calculus textbook does not include a proof of this fact. Because the second-order terms distinguish among the cases in which critical points (points at which $\nabla f = 0$) are maxima, minima or saddles, they play a fundamental role in many areas of science. I found the (simple) derivation of the expansion in the 7th edition of G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, Addison-Wesley (1990), pp. 948–954. I hope this material has not been displaced in any later editions!
because \( \int_0^{2\pi} \cos \theta \, d\theta = \int_0^{2\pi} \sin \theta \, d\theta = 0 \). Among the four second-order terms, the ones involving the cross-partial terms average out to zero because \( \int_0^{2\pi} \sin \theta \cos \theta \, d\theta = \frac{\sin^2 \theta}{2} \bigg|_0^{2\pi} = 0 \). However, the ones involving the “uncrossed partials” do not; in fact, one has

\[
\int_0^{2\pi} \left\{ \cos^2 \theta \right\} \, d\theta = \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta = \pi
\]

and therefore

\[
\int_0^{2\pi} \frac{1}{2!} \left\{ \frac{\partial^2 v(P_0)}{\partial x^2} (a \cos \theta)^2 + \frac{\partial^2 v(P_0)}{\partial y^2} (a \sin \theta)^2 \right\} \, d\theta = \frac{1}{2} \left\{ \frac{\partial^2 v(P_0)}{\partial x^2} a^2 \pi + \frac{\partial^2 v(P_0)}{\partial y^2} a^2 \pi \right\} = \frac{a^2 \pi}{2} \nabla^2 v(P_0).
\]

Putting these computations together, we get

\[
\frac{1}{2\pi} \left[ \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + a \cos \theta, y_0 + a \sin \theta) \, d\theta - v(x_0, y_0) \right] = \frac{1}{2\pi} \int_0^{2\pi} R(a \cos \theta, a \sin \theta) \, d\theta
\]

\[
= a^2 \cdot \left[ \frac{1}{4} \nabla^2 v(P_0) + \frac{1}{2\pi} \int_0^{2\pi} \frac{R(a \cos \theta, a \sin \theta)}{a^2} \, d\theta \right]
\]

The second term on the r. h. side is the circular mean of an expression of the form \( R(\Delta x, \Delta y) \|P - P_0\|^2 \) and can therefore be made as small as we please by taking \( a \) small: in other words, its limit as \( a \to 0^+ \) is zero. The first term does not depend on \( a \) and in fact is simply \( 1/4 \) the value of the Laplacian of \( v \) at \((x_0, y_0)\). If \( v \) is known to be submedian, we can let \( a \to 0^+ \) through values for which it is known that the l. h. side is nonnegative. Thus in the limit we have

\[
0 \leq \frac{1}{4} \nabla^2 v(P_0) \quad \text{for arbitrary } P_0 \in U.
\]

Thus a \( C^2 \) function that satisfies a submedian condition at a point must have a nonnegative Laplacian there, and so a if a function \( v \) is submedian in \( U \) then \( 0 \leq \nabla^2 v \) throughout \( U \).

7. Theorems of Gauss and Liouville. These are two ways in which one can put the (sub)median condition to work to give results that are not at all obvious. We begin with a theorem of Gauss which is really the “equality case” of the submedian condition.

Theorem: Let \( u(x, y) \) be a continuous function defined in a open set \( U \subseteq \mathbb{R}^2 \) with the property that at every point \((x_0, y_0) \in U\), there exist arbitrarily small radii \( a > 0 \) for which the relation

\[
u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) \, d\phi
\]

holds: that is, at each point there are circles of arbitrarily small radius (not necessarily all small circles) centered on the point, such that the average value of \( u \) over the circle equals the value of \( u \) at the center. Then \( u \) is a harmonic function.

Remark: What is remarkable about this theorem is that \( u \) starts out merely continuous (value at each point = limit at that point) and winds up harmonic and therefore infinitely differentiable. Consider the fact that it is possible to give examples of continuous functions that have a derivative at no point of their open domain.}
Proof. Under the hypotheses of this theorem, both the functions \( u(x, y) \) and \(-u(x, y)\) are submedian. Because \( u \) is submedian, it has the property that the inequality

\[
 u(x_0 + r \cos \theta, y_0 + r \sin \theta) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) \, d\phi
\]

holds (in polar coordinates with pole at \((x_0, y_0)\)) in every closed disc with center \((x_0, y_0)\) and radius \(a > 0\) that is entirely contained in \(U\); but the same holds for \(-u\), giving

\[
 \begin{align*}
 -u(x_0 + r \cos \theta, y_0 + r \sin \theta) &\leq -\frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) \, d\phi \\
 u(x_0 + r \cos \theta, y_0 + r \sin \theta) &\geq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) \, d\phi
\end{align*}
\]

for each such disc also. But the two inequalities taken together give

\[
 u(x_0 + r \cos \theta, y_0 + r \sin \theta) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) \, d\phi
\]

identically in each such disc, and that (as we saw in (4) of §5 on p. 7 above) implies that \( u(x, y) \) is harmonic.

We have been working with circular means throughout our discussion of submedian functions, but in fact we could just as well have used area means. If \( v(x, y) \) is submedian in the sense in which we have been using the term, then at every point \((x_0, y_0) \in U\), the relation

\[
 v(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + r \cos \theta, y_0 + r \sin \theta) \, d\theta
\]

holds for every \( r > 0 \) small enough that the closed disc of radius \( r \) and center \((x_0, y_0)\) is entirely contained in \(U\). If we multiply this relation by \( r \, dr \) and integrate with respect to \( r \) from 0 to \( a \) (where \( a > 0 \) is such that the closed disc \( D_a \) of radius \( a \) and center \((x_0, y_0)\) is entirely contained in \(U\), then we find that

\[
 \frac{a^2}{2} v(x_0, y_0) = \int_0^a v(x_0, y_0) \, r \, dr \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^a v(x_0 + r \cos \theta, y_0 + r \sin \theta) \, r \, dr \, d\theta = \frac{1}{2\pi} \int \int_{D_a} v(x, y) \, dA
\]

which says that the value of \( v \) at the center \((x_0, y_0)\) of any closed disc contained in \(U\) is less than or equal to its area mean over that disc. If \( v \) is harmonic, so that both \( v \) and \(-v\) are submedian, then \( v \) is equal to its area mean over any closed disc contained in its domain. On the other hand, by imitating the proof of the weak maximum principle on p. 8 above, one can show quite easily that a function \( v(x, y) \) continuous on \( \overline{U} \), whose value at the center \((x_0, y_0)\) of any closed disc contained in \(U\) is less than or equal to its area mean over that disc, cannot attain a value in the interior that is larger than the largest value it attains on \( \overline{U} \). It follows in the same way that it did for functions whose values were less than or equal to their circular means that functions that are less than or equal to their area means have the property that the inequality

\[
 u(x_0 + r \cos \theta, y_0 + r \sin \theta) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a \cos \phi, y_0 + a \sin \phi) P(r/a, \theta - \phi) \, d\phi
\]

holds (in polar coordinates with pole at \((x_0, y_0)\)) in every closed disc with center \((x_0, y_0)\) and radius \(a > 0\) that is entirely contained in their domains; and that property characterizes submedian functions. So the “circular mean definition” and the “area mean definition” in fact characterize the same class of functions.
Gauss’s theorem basically asserts that a (continuous) function \( u \) such that both \( u \) and \( -u \) satisfy are submedian is a harmonic function. We can therefore characterize harmonic functions as those that are their own area averages over discs:

**Corollary:** Let \( u(x,y) \) be a continuous function defined in an open set \( U \subseteq \mathbb{R}^2 \) with the property that at every point \( (x_0, y_0) \in U \), there exist arbitrarily small radii \( a > 0 \) for which the relation

\[
u(x_0, y_0) = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a u(x_0 + r \cos \theta, y_0 + r \sin \theta) \, r \, dr \, d\theta
\]

holds: that is, at each point there are discs of arbitrarily small radius such that the area mean of \( u \) over the disc equals the value of \( u \) at the center. Then \( u \) is a harmonic function.

We can also use that area-mean characterization to prove

**Theorem [Liouville]:** Let \( u(x,y) \) be a harmonic function defined on all of \( \mathbb{R}^2 \). Suppose that \( u \) is bounded, i.e., that there exists a constant \( M \) such that \( |u(x,y)| \leq M \) holds for all \( (x,y) \in \mathbb{R}^2 \). Then \( u(x,y) \) is a constant function.

**Proof.** We shall show that the value of \( u \) at any point \( P_0 = (x_0, y_0) \) is the same as its value at any other point \( P_1 = (x_1, y_1) \). Let \( \delta = \| P_1 - P_0 \| \), let \( P_m = \frac{P_0 + P_1}{2} \) be their midpoint, and let \( a > \delta \) so that a disc of radius \( a \) centered at either of the points will contain the other. Let \( D_0(a) \) be the disc of radius \( a \) centered on \( P_0 \) and \( D_1(a) \) be the disc of radius \( a \) centered on \( P_1 \). By the area mean property of harmonic functions, we can write

\[
|u(P_1) - u(P_0)| = \frac{1}{\pi a^2} \left| \int_{D_1(a)} u(x,y) \, dA - \int_{D_0(a)} u(x,y) \, dA \right|
\]

but cancellation occurs between the two integrals: the part of each integral taken over the intersection \( D_0(a) \cap D_1(a) \) cancels the corresponding part of the other, so that

\[
\int_{D_1(a)} u(x,y) \, dA - \int_{D_0(a)} u(x,y) \, dA = \int_{D_1(a) \setminus D_0(a)} u(x,y) \, dA - \int_{D_0(a) \setminus D_1(a)} u(x,y) \, dA
\]

and since the integrands are bounded in absolute value by \( M \),

\[
\left| \int_{D_1(a)} u(x,y) \, dA - \int_{D_0(a)} u(x,y) \, dA \right| \leq M \cdot \left( \text{Area (} D_1(a) \setminus D_0(a) \text{) + Area (} D_0(a) \setminus D_1(a) \text{)} \right)
\]

Consider the region \( (D_0(a) \cup D_1(a)) \setminus ((D_0(a) \cap D_1(a)) \). On the one hand, both \( D_0(a) \) and \( D_1(a) \) are contained in the disc of radius \( a + \delta/2 \) centered on \( P_m \), but on the other hand, anything contained in the disc of radius \( a - \delta/2 \) centered on \( P_m \) belongs to both discs. Therefore the area of \( (D_0(a) \cup D_1(a)) \setminus ((D_0(a) \cap D_1(a)) \) is less than or equal to \( \pi(a + \delta/2)^2 - \pi(a - \delta/2)^2 = 2\pi\delta a \). Thus

\[
|u(P_1) - u(P_0)| = \frac{1}{\pi a^2} \left| \int_{D_1(a)} u(x,y) \, dA - \int_{D_0(a)} u(x,y) \, dA \right| \leq \frac{2\pi\delta Ma}{\pi a^2}.
\]

The r. h. side of that relation is a constant multiple of \( \frac{1}{a} \), and because \( u(x,y) \) is defined on the whole plane \( \mathbb{R}^2, a > 0 \) can be as large as we wish so the l. h. side can be as small as we wish. Since the l. h. side of the relation is independent of \( a \), it must be zero, so we have proved that \( u(P_1) = u(P_0) \) for any two points of \( \mathbb{R}^2 \) and thus that \( u(x,y) \) is a constant.\(^{(4)}\)

\(^{(4)}\) With a little massage, one could make the proof we just gave yield the following result: let \( M(a) = \max \{|u(P)| : \|P\| \leq a \} \). Suppose that \( M(a) \) grows “less than linearly fast,” that is, that \( \lim_{a \to \infty} M(a)/a = 0 \). Then the harmonic function \( u(x,y) \) is constant. The best results of this kind take the form: if \( u(x,y) \) is a harmonic function that obeys a one-sided estimate of the form \( u(P) \leq K \cdot \|P\|^\alpha \) (where \( K \) is a constant) then \( u(x,y) \) is a polynomial (in two variables) of degree at most \( n \). The proofs of these results are somewhat more involved; the results themselves play a fundamental rôle in the factorization theory of entire functions of finite order.

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(Remark: While I wanted to give the area-mean proof of Liouville’s theorem,(5) I should point out that by using what we already know about the Poisson kernel, namely the Harnack estimates (see (2) of §5 on p. 6 above), we can prove a stronger result.

Theorem: Let \( u(x, y) \) be a harmonic function defined on all of \( \mathbb{R}^2 \). Suppose that \( u \) is bounded below, i.e., that there exists a constant \( M \) such that \( u(x, y) \geq M \) holds for all \( (x, y) \in \mathbb{R}^2 \). Then \( u(x, y) \) is a constant function.

Proof. With no loss of generality, replacing \( u(x, y) \) by \( u(x, y) - M \) if necessary, we can assume that \( u(x, y) \geq 0 \) holds throughout \( \mathbb{R}^2 \). Using polar coordnates centered at zero, we can write the Harnack estimate

\[
\frac{a - r}{a + r} \leq P(r/a, \theta - \phi) \leq \frac{a + r}{a - r} \quad \text{for } 0 \leq r < a,
\]

multiply it by \( u(a \cos \phi, a \sin \phi) \) preserving the inequalities (because \( u \geq 0 \)), and integrate around the circle of radius \( a \) centered at zero:

\[
\frac{a - r}{a + r} u(0) = \frac{a - r}{a + r} \frac{1}{2\pi} \int_0^{2\pi} u(a \cos \phi, a \sin \phi) \, d\phi
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} P(r/a, \theta - \phi)u(a \cos \phi, a \sin \phi) \, d\phi = u(r \cos \theta, r \sin \theta)
\]

\[
\leq \frac{a + r}{a - r} \frac{1}{2\pi} \int_0^{2\pi} u(a \cos \phi, a \sin \phi) \, d\phi = \frac{a + r}{a - r} u(0).
\]

At any (fixed) point \((r \cos \theta, r \sin \theta) \in \mathbb{R}^2\) we can let \( a \to \infty \) and the inequalities above become

\[
u(0) \leq u(r \cos \theta, r \sin \theta) \leq u(0).
\]

So at every point of \( \mathbb{R}^2 \), the function \( u \) takes the same value that it does at the origin—and is thus constant.}

8. The Schwarz Reflection Principle; Inversion in Circles.(6) Gauss’s theorem enables us to extend the domain of certain harmonic functions by “drawing pictures” rather than by writing analytic expressions. The first case of this operation, however, involves a line—which without real loss of generality we take to be the \( x \)-axis in \( \mathbb{R}^2 \)—rather than a circle. Suppose \( U \subseteq \{(x, y) \in \mathbb{R}^2 : y > 0 \} \) is an open set in the upper half-plane, and denote the set of points in \( \mathbb{R}^2 \) whose coordinates have the form \( \{(x, -y) : (x, y) \in U\} \) by \( U^* \). It is natural to call \( U^* \) the inversion (or reflection) of \( U \) in the \( x \)-axis. Suppose that the boundary \( \partial U \) contains an open line segment \( I = \{(x, 0) : \alpha < x < \beta\} \) of the \( x \)-axis. Then it is easy to verify that the set \( U \cup I \cup U^* \) is an open set in the plane.

Now suppose that \( u(x, y) \) is a continuous function defined on the set \( U \cup I \), such that the restriction of \( u \) to \( U \) is harmonic and such that \( u(x, 0) = 0 \) for each point \((x, 0) \in I\). Then it is easy to see that we can extend \( u \) to be a continuous function on the set \( U \cup I \cup U^* \) by defining the “new \( u \)” (which we shall call \( v \) just to avoid confusion) by setting

\[
v(x, y) = \begin{cases} u(x, y) & \text{if } (x, y) \in U \cup I \\ -u(x, -y) & \text{if } (x, y) \in U^* \cup I. \end{cases}
\]

There is no conflict between the two formulas if \((x, 0) \in I\), because the \( y \)-coordinate is zero there. It is easy to see that this function is continuous, and by Gauss’s mean-value characterization of harmonic

---

(5) The same proof, using volume means rather than area means, works in \( \mathbb{R}^3 \), and there are reasons for not wanting to use the Poisson kernel in that context.

(6) Later, in its three-dimensional case, called “reflection in spheres” by Strauss, pp. 181–186. Many mathematicians use the work “inversion,” however, because the operation is an “inversion” in the transformation sense: if you do it twice, you’re back where you started.
functions, \( v(x, y) \) is a harmonic function both on \( U \)—where we already knew it was harmonic—and on \( U^* \), by symmetry.\(^7\) What is amazing is that \( v(x, y) \) is actually harmonic on the whole set \( U \cup I \cup U^* \). The reason is that it also satisfies the mean-value condition at the points of \( I \): if we install polar coordinates with center at \((x, 0) \in I\), then for every \( a > 0 \) sufficiently small that the closed disc of radius \( a \) and center \((x, 0) \) is contained in \( U \cup I \cup U^* \), we have

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} v(x + a \cos \phi, a \sin \phi) \, d\phi = \frac{1}{2\pi} \left\{ \int_{0}^{\pi} u(x + a \cos \phi, a \sin \phi) \, d\phi + \int_{-\pi}^{0} u(x + a \cos \phi, -a \sin \phi) \, d\phi \right\}
\]

\[
= \frac{1}{2\pi} \left\{ \int_{0}^{\pi} u(x + a \cos \phi, a \sin \phi) \, d\phi - \int_{-\pi}^{0} u(x + a \cos \phi, a \sin \phi) \, d\phi \right\}
= 0 = v(x, 0),
\]

which shows that \( v(x, y) \) also satisfies the mean-value condition at points of \( I \). By Gauss’s theorem, \( v(x, y) \) is harmonic in \( U \cup I \cup U^* \), and we have proved

**Proposition [Schwarz Reflection Principle for Line Segments]:** Suppose \( U \subseteq \{(x, y) \in \mathbb{R}^2 : y > 0\} \) is an open set in the upper half-plane whose boundary \( \partial U \) contains an open line segment \( I = \{(x, 0) : \alpha < x < \beta\} \) of the \( x \)-axis, and let \( U^* = \{(x, -y) : (x, y) \in U\} \), so \( U \cup I \cup U^* \) is an open set in \( \mathbb{R}^2 \). Suppose that \( u(x, y) \) is a continuous function defined on \( U \cup I \), such that the restriction of \( u \) to \( U \) is harmonic and such that \( u(x, 0) = 0 \) for each point \((x, 0) \in I\). Then the function \( v(x, y) \) defined on \( U \cup I \cup U^* \) by

\[
v(x, y) = \begin{cases} 
  u(x, y) & \text{if } (x, y) \in U \cup I \\
  -u(x, -y) & \text{if } (x, y) \in U^* \cup I 
\end{cases}
\]

is harmonic on \( U \cup I \cup U^* \).

Easy extensions of the Schwarz reflection principle are possible. For one example, we nowhere used the connectedness of the interval \( I \) of the \( x \)-axis in the proof, so \( I \) could just as well have been a union of intervals rather than a single interval.\(^8\) One can also weaken the hypothesis that the boundary value where the reflection takes place has to be zero. For one possibility, suppose that the hypothesis that \( u(x, 0) = 0 \) on \( I \) was replaced by the hypothesis that \( u(x, 0) = C \) on \( I \), where \( C \) is a constant. Then the function defined by \( w(x, y) = u(x, y) - C \) would satisfy the hypotheses of the Schwarz reflection principle, so we could extend it to be defined on \( U^* \) by setting \( w(x, y) = -w(x, -y) = -(u(x, -y) - C) = C - u(x, -y) \), so that then \( u(x, -y) = w(x, y) + C = 2C - u(x, -y) \) would extend \( u(x, y) \) to \( U^* \) in such a way that the extension would be harmonic on \( C \cup I \cup U^* \). One has to be a bit careful with this trick, however, since it can have unexpected results. One way to choose the usual polar angle \( \theta \) in the plane (which is a harmonic function: it almost-trivially satisfies the Laplace equation in polar coordinates) would make it \( 0 \) on the positive \( x \)-axis, make it just a bit smaller than \( \pi \) just above the negative \( x \)-axis and just a bit larger than \( -\pi \) just below the negative \( x \)-axis. This function, since it takes the value \( 0 \) on the positive \( x \)-axis, simply reflects across the positive \( x \)-axis to give itself. However, if we regarded this function as being defined on the upper half-plane only, we could easily extend it to the upper half-plane united with the negative \( x \)-axis in such a way as to be continuous: just let it equal \( \pi \) on the negative \( x \)-axis. If we reflected this function across the negative \( x \)-axis according to the recipe we just gave, we would set \( \theta(x, y) = 2\pi - \theta(x, -y) \) in the lower half-plane. For points \((x, y)\) just below the positive \( x \)-axis, this function would give us values very close to \( 2\pi - 0 = 2\pi \). Now there is nothing wrong with letting those points have polar angles near \( 2\pi \), but those choices do not agree with the choices we made by reflecting over the positive \( x \)-axis the same polar-angle function defined in the upper half-plane that we just reflected over the negative \( x \)-axis. There is no contradiction here: the constant value it took on the negative \( x \)-axis was different from the constant value it took on the positive \( x \)-axis, and our extension method breaks down if we have to use “two values for the constant \( C \)” instead of one; but one has to think critically about what one is doing when one reflects across a boundary at which the value of the reflected function is not zero.

\(^7\) You can also check this by observing that replacing \( y \) by \(-y \) does not change second derivatives.

\(^8\) If you know the topology of \( \mathbb{R} \), you know that this means that one can reflect across any (nonempty relatively) open subset of the \( x \)-axis.