Accelerated sequences usually converge faster

Theorem 2.13 on page 88 of Burden and Faires is incorrect as stated in the text. We provide a corrected version and examples here.

Recall that if p_n is a sequence which converges to p linearly, we define a new sequence $\hat{p}_n$ by the formula

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

For example, if $p_n = 1/n$ we have

$$\hat{p}_n = 1/n - (1/(n+1) - 1/n)/(1/(n+2) - 2/(n+1) + 1/n = 1/(2(n+1))$$

which converges to 0, but not much faster than the original sequence. The problem is that for this sequence it converges linearly, but $\lim_{n\to\infty}(p_{n+1}/p_n) = 1$. The modification of the theorem in the book which is true is the following.

Theorem: (corrected version of 2.3). Let $\{p_n\}$ be a sequence that converges linearly to a limit p and such that $\lim_{n\to\infty}(p_{n+1} - p)/(p_n - p) = \lambda < 1$. Suppose for large n we have $(p_{n+1} - p)(p_n - p) > 0$. Then the accelerated sequence $\{\hat{p}_n\}$ defined above converges faster than the original sequence in the sense that $\lim_{n\to\infty}(\hat{p}_n - p)/(p_n - p) = 0$.

Proof: We follow the suggestion of problem 14 of section 2.5. Write

$$(p_{n+1} - p)/(p_n - p) = \lambda + \delta_n.$$

Then by assumption that $\lim_{n\to\infty}(p_{n+1}/p_n) = \lambda$ we have $\lim_{n\to\infty}\delta_n = 0$. The assumption on the sign in the theorem shows that for large n $(p_{n+1} - p)/(p_n - p) = |(p_{n+1} - p)/(p_n - p)|$.

We have that $(p_{n+1} - p) = (p_n - p)(\lambda + \delta_n)$, so that $(p_{n+2} - p) = (p_{n+1} - p)(\lambda + \delta_{n+1}) = (p_n - p)(\lambda + \delta_n)(\lambda + \delta_{n+1})$. Using this we can write

$$p_{n+1} - p_n = (p_{n+1} - p) - (p - p_n) = (\lambda + \delta_n - 1)(p_n - p)$$

and

$$p_{n+2} - 2p_{n+1} + p_n = (p_{n+2} - p) - 2(p_{n+1} - p) + (p_n - p) =$$
$$((\lambda + \delta_n)(\lambda + \delta_{n+1}) - 2(\lambda + \delta_n) + 1)(p_n - p)$$

Using these we get

$$\hat{p}_n - p = p_n - p - \frac{(\lambda + \delta_n - 1)^2(p_n - p)^2}{((\lambda + \delta_n)(\lambda + \delta_{n+1}) - 2(\lambda + \delta_n) + 1)(p_n - p)}$$

and dividing both sides by $p_n - p$ gives that

$$\frac{\hat{p}_n - p}{p_n - p} = 1 - \frac{(\lambda + \delta_n - 1)^2}{((\lambda + \delta_n)(\lambda + \delta_{n+1}) - 2(\lambda + \delta_n) + 1)}$$

As n approaches ∞ the denominator of the right hand side approaches $\lambda^2 - 2\lambda + 1$ which is not 0 since $|\lambda| < 1$. Thus we take the limit of the numerator and denominator and divide, to give that

$$\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 1 - (\lambda - 1)^2/(\lambda - 2\lambda + 1) = 1 - 1 = 0$$

as claimed in the theorem.

Example: The usual application of the theorem is to accelerate the linear converge of a fixed point iteration. Since we are usually in the situation where the derivative of the function iterated is bounded above by a constant $k < 1$, we have linear convergence with the λ of the theorem satisfying $\lambda \leq k < 1$ so the accellerated series actually converges faster.

For a simple example of this, consider $g(x) = \pi + .999999 \sin(x)$ which has a fixed point at π. Since the sin function is between -1 and 1, the interval [2,4] is mapped to itself by $g(x)$, so the fixed point iteration theorem tells us that taking $p_0 = 3$ and defining $p_{N=1} = g(p_n)$ will give a sequence converging to π. The problem is that it converges very slowly. In fact, taking $k = .999999$ even after a million iterations we have $k^{10^6} = 0.367$ so that we do not get much accuracy. If we take 100000 iterations we get $p_{100000} = 3.1363...$, not a great approximation to π. If instead we accelerate the convergence by taking $p_0, p_1 = g(p_0), p_2 = g(p_1), \hat{p}_0 = p_2 - (p_1 - p_0)^2/(p_2 - 2p_2 + p_0)$ and iterating by replacing p_0 by $\hat{p}_0$ and continuing this scheme (Steffensen's method) we obtain the sequence

$$q_0 = 3.141591866602674158623327009$$

$$q_1 = 3.141592653589793238462643342$$

which already agrees with π to more than 15 decimal places.