Math 373 — Spring 2000

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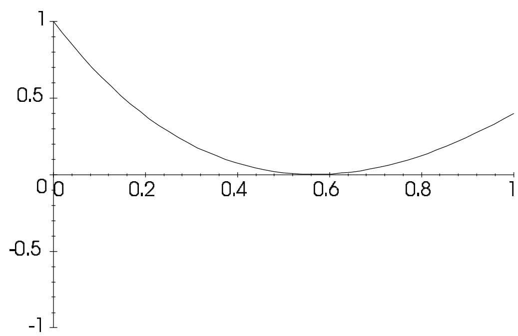
Answers to Homework 6 (lecture 6 – Due 2/10/00)

The odd numbered exercises have answers in the book, so you can check your work. Only a few will be answered here. Even numbered ones are answered here.

Exercise 1 (Page 86: 1a, 2) *Let* $f(x) = x^2 - 2xe^{-x} + e^{-2x}$. *Use Newton's Method to find a solution to within* 10^{-5} *to*
$f(x) = 0$ *on* $[0, 1]$.

Let us see what the graph of f looks like.



That sure looks very much like a multiple root from the graph. We compute $f'(x) = 2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x}$. Set $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 2xe^{-x} + e^{-2x}}{2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x}}$. From the graph we choose to start at $p_0 = 1$. $p_1 = g(p_0) = .76894142$.

$p_2 = g(p_1) = g(.76894142) = .66458979$. $p_3 = g(p_2) = g(.66458979) = .61503324$.
$p_4 = g(p_3) = g(.61503324) = .59088382$. $p_5 = g(p_4) = g(.59088382) = .57896293$.
$p_6 = g(p_5) = g(.57896293) = .57304052$. $p_7 = g(p_6) = g(.57304052) = .57008876$.
$p_8 = g(p_7) = g(.57008876) = .56861524$. $p_7 = g(p_6) = g(.56861524) = .56787907$.
$p_{10} = g(p_9) = g(.56787907) = .56751114$. $p_{11} = g(p_{10}) = g(.56751114) = .56732719$.
$p_{12} = g(p_{11}) = g(.56732719) = .56723541$. $p_{13} = g(p_{12}) = g(.56723541) = .56718922$.
$p_{14} = g(p_{13}) = g(.56718922) = .56716573$. $p_{15} = g(p_{14}) = g(.56716573) = .56715484$.
$p_{16} = g(p_{15}) = g(.56715484) = .56714779$.

Notice that $|p_{16} - p_{15}| = |.56714779 - .56715484| = 7.05 \times 10^{-6} < 1 \times 10^{-5}$ so we will stop here and return the approximate root $.56714779$.

We now do the modified form.

Set $h(x) = x - \frac{f(x)f'(x)}{(f'(x))^2 - f(x)f''(x)} = \frac{x^2e^{-x}+x^3e^{-x}-x^2e^{-2x}-xe^{-3x}+x^2e^{-3x}+x^4e^{-x}-2x^3e^{-2x}-2xe^{-2x}+e^{-3x}+e^{-4x}}{x^2-2xe^{-x}+2x^2e^{-x}-4xe^{-2x}+e^{-2x}+2e^{-3x}+xe^{-3x}+x^3e^{-x}-2x^2e^{-2x}}$. Again I will start at $p_0 = 1$.

$p_1 = h(p_0) = h(1.) = .58896\,703$. $p_2 = g_1(.58896\,703) = .56722\,766$.

$p_3 = h(p_2) = h(.56722\,766) = .56714\,311$. $p_4 = h(.56714\,311) = .56714\,293$.

We are already at the root to within our given tolerance. Return the approximate root .56714293. Speed is clearly improved, and the approximation looks better also since the original approximations were decreasing.

Now look at one other of the functions. Let $F(x) = \cos\left(x+\sqrt{2}\right) + x\left(\frac{x}{2}+\sqrt{2}\right)$. Use modified Newton Raphson to find a root of F in $[-2,-1]$. Set

$$\begin{aligned} k(x) &= x - \frac{F(x)F'(x)}{(F'(x))^2 - F(x)F''(x)} \\ &= x - \left(\cos\left(x+\sqrt{2}\right) + x\left(\frac{1}{2}x+\sqrt{2}\right)\right) \\ &\quad \times \frac{-\sin\left(x+\sqrt{2}\right)+x+\sqrt{2}}{\left(-\sin\left(x+\sqrt{2}\right)+x+\sqrt{2}\right)^2 - \left(\cos\left(x+\sqrt{2}\right)+x\left(\frac{1}{2}x+\sqrt{2}\right)\right)\left(-\cos\left(x+\sqrt{2}\right)+1\right)} \end{aligned}$$

This one we will just start at one of the endpoints, iterating a few times to see if we indeed get convergence.

$p_0 = -1.$ $p_1 = k(-1.) = -1.4118537$. $p_2 = k(-1.4118537) = -1.4111344$.

$p_3 = k(-1.4111344) = -1.4100741$. $p_4 = k(-1.4100741) = -1.41007\,41$.

We have converged but not to what might be expected from the answer to $1.a$. Now we will increase precision from 10 digit arithmetic with 8 displayed and carried over into the next computation to 15 digits with 12 displayed and carried over.

$p_0 = -1.$ $p_1 = k(-1.) = -1.41185332824$. $p_2 = k(-1.41185332824) = -1.41419159001$. $p_3 = k(-1.41419159001) = -1.41419\,159001$.

It is still off, but it is closer. Round-off error has been significant. Let us try 20 digits.

$p_0 = -1.$ $p_1 = k(-1.) = -1.41185\,33282\,425713365$. $p_2 = k(-1.41185\,33282\,425713365) = -1.41421\,35622\,649365404$.

$p_3 = k(-1.41421\,35622\,649365404)$ gives a division by 0 error using Maple. This is a dramatic illustration of what can happen in blindly applying a method. Maple computes $F(-1.41421\,35622\,649365404) = 0$, $F'(-1.41421\,35622\,649365404) = 0$, $F''(-1.41421\,35622\,649365404) = 1.0 \times 10^{-20}$.

Exercise 2 (Page 90: 10c) *Use Steffensen's method to approximate a solution to* $3x^2 - e^x = 0$ *to within* 10^{-5}.

Let $g(x) = \frac{e^x}{3x}$. The fixed points of g are precisely the zeros of $3x^2 - e^x$. $g'(x) = \frac{1}{3}\frac{e^x}{x} - \frac{1}{3}\frac{e^x}{x^2} = \frac{1}{3}e^x\frac{x-1}{x^2}$ is negative for $0 < x < 1$ and positive for $1 < x$. If one starts iterating g to the right of 1 the iterates will diverge to ∞. If one starts the iteration below 1 the iterates approach 0. So we try a different g, say $g(x) = \sqrt{e^x/3}$. Then $g'(x) = \frac{1}{6}\sqrt{3}e^{\frac{1}{2}x}$ is always positive, and $g'(0) = .28867\,513$, $g'(1) = .47594484$ so g takes the interval $[0,1]$ to itself. $g''(x) = \frac{1}{12}\sqrt{3}e^{\frac{1}{2}x}$ is also always positive, so the maximum value of g' on $[0,1]$ occurs at 1. $g'(1.) = .47594\,484 < 1$ so fixed point iteration will converge.

Set

$$p_0^{(0)} = 0, \quad p_1^{(0)} = g(0) = .57735027, \quad p_2^{(0)} = g(.57735027) = .7705652,$$

$$
\begin{aligned}
p_0^{(1)} &= p_0^{(0)} - \left(p_1^{(0)} - p_0^{(0)}\right)^2 \Big/ \left(p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}\right) \\
&= 0 - (.57735027 - 0)^2 / (.7705652 - 2 \cdot .57735027 + 0) = .86774972. \\
p_1^{(1)} &= g\left(p_0^{(1)}\right) = g\,(.86774972) = .89098179, \quad p_2^{(1)} = g\,(.89098179) = .90139181. \\
p_0^{(2)} &= p_0^{(1)} - \left(p_1^{(1)} - p_0^{(1)}\right)^2 \Big/ \left(p_2^{(1)} - 2p_1^{(1)} + p_0^{(1)}\right) \\
&= .86774972 - (.89098179 - .86774972)^2 / (.90139181 - 2 \cdot .89098179 + .86774972) = .90984354. \\
p_1^{(2)} &= g\left(p_0^{(2)}\right) = g\,(.90984354) = .90993294, \quad p_2^{(2)} = g\left(p_1^{(2)}\right) = g\,(.90993294) = .90997361. \\
p_0^{(3)} &= p_0^{(2)} - \left(p_1^{(2)} - p_0^{(2)}\right)^2 \Big/ \left(p_2^{(2)} - 2p_1^{(2)} + p_0^{(2)}\right) \\
&= .90984354 - (.90993294 - .90984354)^2 / (.90997361 - 2 \cdot .90993294 + .90984354) = .91000755. \\
p_1^{(3)} &= g\left(p_0^{(3)}\right) = g\,(.91000755) = .91000756, \quad p_2^{(3)} = g\left(p_1^{(3)}\right) = g\,(.91000756) = .91000757
\end{aligned}
$$

You can stop here as all of the three iterates with superscript 3 agree except in the last decimal place. This looks like an invitation to a divide by 0 error to continue, so return the approximate root .91000757.