Math 373 — Spring 2000

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Answers to Homework 4 (lecture 4 – Due 2/3/00)

The odd numbered exercises have answers in the book, so you can check your work. Only a few will be answered here. Even numbered ones are answered here.

Exercise 1 (Page 63: 4) *The following four methods are proposed to compute* $7^{1/5}$*. Rank them in order, based on their apparent speed of convergence, assuming* $p_0 = 1$*.*

(a) $p_n = \left(1 + \frac{7-p_{n-1}^3}{p_{n-1}^2}\right)^{1/2}$. Let $g_1(x) = \left(1.0 + \frac{7.0-x^3}{x^2}\right)^{1/2}$ and $p_0 = 1.0$. First check that $x = \left(1.0 + \frac{7.0-x^3}{x^2}\right)^{1/2}$ if and only if $x > 0$ and $x^2 = 1.0 + \frac{7.0-x^3}{x^2}$ if and only if $x^4 - x^2 - 7.0 + x^3 = 0$ and $x > 0$. $7^{1/2}$ is not a solution of this equation. We'll compute anyway.:
$p_1 = g_1(1) = 2.6457513,$
$p_2 = g_1(2.6457513) = 0.80358652i.$
This is not a real valued function.

(b) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{p_{n-1}^2}$. Let $g_2(x) = x - \frac{x^5-7.0}{x^2}$ and $p_0 = 1.0$. By observation, $g(p) = p$ if and only if $p^5 - 7 = 0$. Now iterate:
$p_1 = g_2(1) = 7.0,$
$p_2 = g_2(7.0) = -335.85714,$
$p_3 = g_2(-335.85714) = 3.7884356 \times 10^7.$
This one does not seem to converge at all.

(c) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{5p_{n-1}^4}$. This is Newton's method. We expect fast convergence. Let $g_3(x) = x - \frac{x^5-7.0}{5x^4}$ and $p_0 = 1$. Then:
$p_1 = g_3(1) = 2.2, \quad p_2 = g_3(2.2) = 1.8197637, \quad p_3 = g_3(1.8197637) = 1.5834748,$
$p_4 = g_3(1.5834748) = 1.489461, \quad p_5 = g_3(1.489461) = 1.4760224, p_6 = g_3(1.4760224) = 1.4757732,$
$p_7 = g_3(1.4757732) = 1.4757732.$
To the precision of the arithmetic being used, this has converged.

(d) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{12}$. Let $g_4(x) = x - \frac{x^5-7.0}{12}$. Again a fixed point of g_4 is a root of $x^5 - 7$. Let $p_0 = 1$. Then:
$p_1 = g_4(1) = 1.5, \quad p_2 = g_4(1.5) = 1.4505208, \quad p_3 = g_4(1.4505208) = 1.4987497,$

$p_4 = g_4(1.4987497) = 1.4519035$, $p_5 = g_4(1.4519035) = 1.4975771$, $p_6 = g_4(1.4975771) = 1.4531923$.

This looks like it is converging, but it is not there yet.

Hence, to answer the question, ()c appears to be converging fastest, then ()d, and the other two do not appear to converge.

Exercise 2 (Page 63: 6) *Use a fixed point iteration method to determine a solution accurate to within* 10^{-3} *for* $x^3 - x - 1 = 0$ *in* $[1, 2]$*. Use* $p_0 = 1$*.*

Let us try $g(x) = (x + 1.0)^{1/3}$. $g(1) = \sqrt[3]{2} = 1.2599211$, and $g(2) = \sqrt[3]{3} = 1.4422496$ so at least the end points of $[1, 2]$ are taken into that interval. $g'(x) = \frac{1}{3\left(\sqrt[3]{(x+1)}\right)^2}$ is always positive on $[1, 2]$ so g is an increasing function and the entire interval must be taken into itself. Moreover, on $[1, 2]$, g' is clearly decreasing as the denominator increases and the numerator stays constant. Then its largest value is at 1, where $g'(1) = \frac{1}{6}\sqrt[3]{2} < \frac{1}{3}$. Fixed point iteration should work reasonably fast.

$p_1 = g(1) = 1.2599211$, $p_2 = g(1.2599211) = 1.3122938$, $p_3 = g(1.3122938) = 1.3223538$,
$p_4 = g(1.3223538) = 1.3242687$, $p_5 = g(1.3242687) = 1.3246326$, $p_6 = g(1.3246326) = 1.3247017$,
$p_7 = g(1.3247017) = 1.3247149$, $p_8 = g(1.3247149) = 1.3247174$, $p_9 = g(1.3247174) = 1.3247179$.

We are now at the last digit in the precision used. The sequence has effectively converged.

Exercise 3 (Page 64: 12) *For each of the following equations, determine a function* g *and an interval* $[a, b]$ *on which fixed point iteration will converge to a positive solution of the equation. Find the solution to within* 10^{-5}*.*

(a) $3x^2 - e^x = 0$. The function $f(x) = 3x^2 - e^x$ is negative at 0 and positive at 1 so we will try the interval $[0, 1]$. A reasonable candidate as a function to iterate is $g(x) = \sqrt{e^x/3}$. It takes $[0, 1]$ to itself by observation, since it is an increasing function and the endpoints go into the interval. Moreover, $g'(x) = \frac{1}{6}\sqrt{3}e^{\frac{1}{2}x} < \frac{\sqrt{3e}}{6} = .47594484$ on $[0, 1]$ since an increasing function takes on its maximum at the right hand endpoint. Let us start where the derivative is smaller, namely at $p_0 = 0$.

$p_1 = g(0.) = .57735027$, $p_2 = g(.57735027) = .7705652$, $p_3 = g(.7705652) = .84872204$,

$p_4 = g(.84872204) = .88254533$, $p_5 = g(.88254533) = .89759754$, $p_6 = g(.89759754) = .90437844$,

$p_7 = g(.90437844) = .9074499$, $p_8 = g(.9074499) = .90884457$, $p_9 = g(.90884457) = .90947856$,

$p_{10} = g(.90947856) = .9097669$, $p_{11} = g(.9097669) = .90989807$, $p_{12} = g(.90989807) = .90995775$,

$p_{13} = g(.90995775) = .9099849$, $p_{14} = g(.9099849) = .90999726$, $p_{15} = g(.90999726) = .91000288$,

$p_{16} = g(.91000288) = .91000544$, $p_{17} = g(.91000544) = .9100066$, $p_{18} = g(.9100066) = .91000713$.

Although it is still possible that small changes will add up to effect the 5^{th} decimal place, we suspect that they will not, and since g the iterates have been increasing, I would give the approximate root as .91001. It is certainly more than .910007and highly unlikely to reach .91002.

Clearly 1 would have been a better starting point. If $p_0 = 1$,

$p_1 = g(1.) = .95188\,967$, $p_2 = g(.95188\,967) = .92926502$, $p_3 = g(.92926502) = .9188121$,

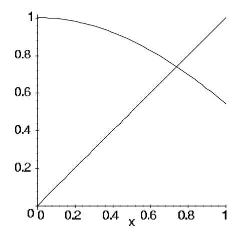
$p_4 = g(.9188121) = .91402\,25$, $p_5 = g(.91402\,25) = .91183\,621$, $p_6 = g(.91183\,621) = .91083999$,

$p_7 = g(.91083999) = .9103864$, $p_8 = g(.9103864) = .91017996$, $p_9 = g(.91017996) = .91008601$,

$p_{10} = g(.91008601) = .91004\,326$, $p_{11} = g(.91004\,326) = .91002381$, $p_{12} = g(.91002381) = .91001496$,

$p_{13} = g(.91001496) = .91001093$, $p_{14} = g(.91001093) = .91000\,91$, $p_{14} = g(.91000\,91) = .91000\,827$, $p_{15} = g(.91000\,827) = .91000789$

(b) $x - \cos(x) = 0$. The function $g(x) = \cos(x)$ takes $[0,1]$ to itself and has the absolute value of its derivative bounded by $\sin(1) < 1$. So let us try using it . From the graph below, we can start with $p_0 = 0.8$.



$p_1 = \cos(.8) = .69670\,671$, $p_2 = \cos(.69670\,67) = .76695\,964$, $p_3 = \cos(.76695\,964) = .72002385$,

$p_4 = \cos(.72002385) = .75179$, $p_5 = \cos(.75179) = .73046756$, $p_6 = \cos(.73046756) = .74486252$,

$p_7 = \cos(.74486252) = .73518\,11$, $p_8 = \cos(.73518\,1) = .74170937$, $p_9 = \cos(.74170937) = .73731487$,

$p_{10} = \cos(.73731487) = .74027645$, $p_{11} = \cos(.7402764) = .73828216$, $p_{12} = \cos(.7382821) = .73962583$,

$p_{13} = \cos(.73962583) = .73872081$, $p_{14} = \cos(.73872081) = .7393305$, $p_{15} = \cos(.7393305) = .73891983$,

$p_{16} = \cos(.73891983) = .73919\,647$, $p_{17} = \cos(.73919\,647) = .73901013$, $p_{18} = \cos(.7390101) = .73913567$,

$p_{19} = \cos(.73913567) = .73905109$, $p_{20} = \cos(.73905109) = .73910806$, $p_{21} = \cos(.73910806) = .73906\,969$,

$p_{22} = \cos(.73906\,969) = .73909554$, $p_{23} = \cos(.73909554) = .73907812$, $p_{24} = \cos(.73907812) = .73908986$,

$p_{25} = \cos(.73908986) = .73908195$.

There have been enough iterates close to our fixed point, and they have alternatively over and under estimated, so that I would give the answer as the fixed point p is .73908 or perhaps .73909.