Math 373 — Spring 2000

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Answers to Homework 2 (lecture 2 – Due 1/22/00)

The odd numbered exercises have answers in the book, so you can check your work. Only a few will be answered here. Even numbered ones are answered here.

Exercise 1 (Page 37: 1) *(a). Use three-digit chopping arithmetic to compute the sum* $\sum_{1=1}^{10} \dfrac{1}{i^2}$ *first by adding from* $i = 1$ *to 10 and then by adding from* $i = 10$ *to 1. Which is more accurate and why?*

Adding down should be more accurate as the numbers you are adding at any time are closer to each other at each step, and there is a smaller round off error. Using 3 digits we get $\frac{1}{1} = 1.00$, $\frac{1}{4} = .250$, $\frac{1}{9} = .111$, $\frac{1}{16} = .0625$, $\frac{1}{25} = 4.00, \times 10^{-2}$ $\frac{1}{36} = 2.77 \times 10^{-2}$, $\frac{1}{49} = 2.040 \times 10^{-2}$, $\frac{1}{64} = 1.56 \times 10^{-2}$, $\frac{1}{81} = 1.23 \times 10^{-2}$, $\frac{1}{100} = 1.00 \times 10^{-2}$.

Then $1 + .25 = 1.25$, $1.25 + .111 = 1.36$, $1.36 + .0625 = 1.42$, $1.42 + .0400 = 1.82$, $1.82 + .0277 = 1.84$, $1.84 + .0204 = 1.86$, $1.86 + .0156 = 1.87$, $1.87 + .0123 = 1.88$, $1.88 + .0100 = 1.89$ and $.0100 + .0123 = .0223$, $.0223 + .0156 = .0379$, $.0379 + .0204 = .0583$, $.0583 + .0277 = .0860$, $.0860 + .0400 = .126$, $.126 + .0625 = .188$, $.188 + .111 = .299$, $.299 + .250 = .549$, $.549 + 1 = 1.54$.

Note $\sum_{i=1}^{10} \frac{1}{i^2} = \frac{1968329}{1270080} \approx 1.5498$.

Exercise 2 (Page 38: 3) *The Maclauren series for the arctangent function converges for* $-1 < x < 1$ *and is given by*

$$\arctan(x) = \sum_{i=1}^{n} (-1)^{i+1} \frac{x^{2i-1}}{(2i-1)}.$$

Use the fact that $\tan\left(\frac{\pi}{4}\right) = 1$ *to determine the number of terms of this series we must sun to obtain an approximation of* π *within a tolerance of* ε.

Four times this series evaluated at 1 will converge to π. That is, the series $\sum_{i=1}^{\infty} (-1)^i (4) \frac{1}{2i-1}$ converges to π. This is an alternating series (check your calculus 2 book for this if you need to review this). The absolute error in using n terms of an alternating series to approximate the sum is at most the absolute value of the $(n+1)^{st}$ term. Then $\left|\sum_{i=1}^{n} (-1)^{i+1} (4) \frac{1}{(2i-1)} - \pi\right| \le \frac{4}{(2n+1)}$, and this is at most ε if we take $4 \le (2n+1)\varepsilon$, $\frac{\frac{4}{\varepsilon} - 1}{2} \le n$. If $\varepsilon = 10^{-3}$, then take $\left(\frac{4000-1}{2}\right) = 1999.5 \le n$, or $2000 \le n$.

Exercise 3 (Page 38: 4) *The problem observes that* $\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$. *Find the number of terms to sum to be sure to get* π *within* 10^{-3}.

Using the previous Maclauren series for the arctangent function, we get

$$\pi = \sum_{i=1}^{\infty} (-1)^{i+1} (4) \frac{(1/2)^{2i-1} + (1/3)^{2i-1}}{(2i-1)}$$

since series add termwise within their intervals of convergence. As above, the truncation error from using a sum of n terms in place of this alternating series is at most the absolute value of the $(n+1)^{st}$ term . A very convenient approximation to know is $2^{10} \gtrapprox 10^3$ and so $2^{-11} < .5 \times 10^{-3}$. Also $2 < 3$ so $2^{-i} > 3^{-i}$ and $3^{-11} < .5 \times 10^{-3}$. Since $2 \cdot 11 - 1 > 4$, the absolute value of the 6^{th} term is at most $4 \cdot 10^{-3}/21 < 10^{-3}$.

We (or rather Maple V) compute that $\sum_{i=1}^{6} (-1)^{i+1} (4) \frac{(1/2)^{2i-1} + (1/3)^{2i-1}}{(2i-1)} = 3.1415616$.

Exercise 4 (Page 38: 6) *Find the rates of convergence of the following sequences as* $n \longrightarrow \infty$.

(a) $\lim_{n\to\infty} \sin(1/n) = 0$. Since $\lim_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = 1$, the rate of convergence of the sequence is $O\left(\frac{1}{n}\right)$.

(b) $\lim_{n\to\infty} \sin(1/n^2) = 0$. Since $\lim_{n\to\infty} \dfrac{\sin(1/n^2)}{1/n^2} = 1$, the rate of convergence of the sequence is $O\left(\frac{1}{n^2}\right)$.

(c) $\lim_{n\to\infty} (\sin(1/n))^2 = 0$. Since $\lim_{n\to\infty} \dfrac{(\sin(1/n))^2}{(1/n)^2} = 1$, the rate of convergence of the sequence is $O\left(\frac{1}{n^2}\right)$.

(d) $\lim_{n\to\infty} (\ln(n+1) - \ln(n)) = 0$. Now $\ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right)$. By l'Hôpital's rule,

$$\lim_{n\to\infty} \left(\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\right) = \lim_{n\to\infty} \left(\frac{\left(1+\frac{1}{n}\right)^{-1}\left(\frac{1}{n}\right)^2}{\left(\frac{1}{n}\right)^2}\right) = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right) = 1$$

so this converges at a rate $O\left(\frac{1}{n}\right)$.