Math 373 — Spring 2000
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Answers to Homework 2 (lecture 2 — Due 1/22/00)

The odd numbered exercises have answers in the book, so you can check your work. Only a few will be answered here. Even numbered ones are answered here.

Exercise 1 (Page 37: 1) (a). Use three-digit chopping arithmetic to compute the sum \( \sum_{i=1}^{10} \frac{1}{i^2} \) first by adding from \( i = 1 \) to 10 and then by adding from \( i = 10 \) to 1. Which is more accurate and why?

Adding down should be more accurate as the numbers you are adding at any time are closer to each other at each step, and there is a smaller round off error. Using 3 digits we get

\[
\frac{1}{1} = 1.00, \quad \frac{1}{4} = .250, \quad \frac{1}{9} = .111, \quad \frac{1}{16} = .0625, \quad \frac{1}{25} = 4.00 \times 10^{-2}, \quad \frac{1}{36} = 2.77 \times 10^{-2}, \quad \frac{1}{49} = 2.040 \times 10^{-2}, \quad \frac{1}{64} = 1.56 \times 10^{-2}, \quad \frac{1}{81} = 1.23 \times 10^{-2}, \quad \frac{1}{100} = 1.00 \times 10^{-2}.
\]

Then \( 1 + .25 + .111 = 1.36, \quad 1.36 + .0625 = 1.42, \quad 1.42 + .0400 = 1.82, \quad 1.82 + .0277 = 1.84, \quad 1.84 + .0204 = 1.86, \quad 1.86 + .0156 = 1.87, \quad 1.87 + .0123 = 1.88, \quad 1.88 + .0100 = 1.89 \) and

\[
.0100 + .0123 = .0223, \quad .223 + .0156 = .379, \quad .0379 + .0204 = .0583, \quad .0583 + .0277 = .0860, \quad .0860 + .0400 = .126, \quad .126 + .0625 = .188, \quad .188 + .111 = .299, \quad .299 + .250 = .549, \quad .549 + 1 = 1.549.
\]

Note \( \sum_{i=1}^{10} \frac{1}{i^2} = \frac{163829}{120000} \approx 1.3498. \)

Exercise 2 (Page 38: 3) The Maclaurin series for the arctangent function converges for \( -1 < x < 1 \) and is given by

\[
\arctan(x) = \sum_{i=1}^{\infty} (-1)^i \frac{x^{2i-1}}{(2i-1)}.
\]

Use the fact that \( \tan \left( \frac{\pi}{2} \right) = 1 \) to determine the number of terms of this series we must sum to obtain an approximation of \( \pi \) within a tolerance of \( \varepsilon. \)

Four times this series evaluated at 1 will converge to \( \pi. \) That is, the series \( \sum_{i=1}^{\infty} (-1)^i \frac{1}{(2i-1)} \) converges to \( \pi. \) This is an alternating series (check your calculus book for this if you need to review this). The absolute error in using \( n \) terms of an alternating series to approximate the sum is at most the absolute value of the \( (n+1)st \) term. Then

\[
\left| \sum_{i=1}^{n} (-1)^i \frac{1}{(2i-1)} - \pi \right| \leq \frac{1}{(2n+1)},
\]

and this is at most \( \varepsilon \) if we take \( 4 \leq (2n+1) \varepsilon, \quad \frac{4}{2} - 1 \leq n. \) If \( \varepsilon = 10^{-3} \), then take \( \frac{4000 - 1}{2} = 1999.5 \leq n, \) or \( 2000 \leq n. \)
Exercise 3 (Page 38: 4)  The problem observes that \( \frac{\pi}{4} = \arctan \left( \frac{1}{2} \right) + \arctan \left( \frac{1}{3} \right) \). Find the number of terms to sum to be sure to get \( \pi \) within \( 10^{-3} \).

Using the previous Maclaurin series for the arctangent function, we get

\[
\pi = \sum_{i=1}^{\infty} (-1)^{i+1} \left[ \frac{(1/2)^{2i-1}}{(2i-1)} + \frac{(1/3)^{2i-1}}{(2i-1)} \right]
\]

since series add termwise within their intervals of convergence. As above, the truncation error from using a sum of \( n \) terms in place of this alternating series is at most the absolute value of the \( (n+1) \)st term. A very convenient approximation to know is \( 2^{10} \approx 10^3 \) and so \( 2^{-11} < .5 \times 10^{-3} \). Also \( 2 < 3 \) so \( 2^{-i} > 3^{-i} \) and \( 3^{-11} < .5 \times 10^{-3} \). Since \( 2 \cdot 11 - 1 > 4 \), the absolute value of the \( 6 \)th term is at most \( 4 \cdot 10^{-3} / 21 < 10^{-3} \).

We (or rather Maple V) compute that \( \sum_{i=1}^{6} (-1)^{i+1} \left[ \frac{(1/2)^{2i-1}}{(2i-1)} + \frac{(1/3)^{2i-1}}{(2i-1)} \right] = 3.1415616 \).

Exercise 4 (Page 38: 6)  Find the rates of convergence of the following sequences as \( n \to \infty \).

(a) \( \lim_{n \to \infty} \sin \left( \frac{1}{n} \right) = 0 \). Since \( \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{1/n} = 1 \), the rate of convergence of the sequence is \( O \left( \frac{1}{n} \right) \).

(b) \( \lim_{n \to \infty} \sin \left( \frac{1}{n^2} \right) = 0 \). Since \( \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n^2} \right)}{1/n^2} = 1 \), the rate of convergence of the sequence is \( O \left( \frac{1}{n^2} \right) \).

(c) \( \lim_{n \to \infty} \left( \sin \left( \frac{1}{n} \right) \right)^2 = 0 \). Since \( \lim_{n \to \infty} \frac{\left( \sin \left( \frac{1}{n} \right) \right)^2}{(1/n)^2} = 1 \), the rate of convergence of the sequence is \( O \left( \frac{1}{n^2} \right) \).

(d) \( \lim_{n \to \infty} \left( \ln \left( n + 1 \right) - \ln \left( n \right) \right) = 0 \). Now \( \ln \left( n + 1 \right) - \ln \left( n \right) = \ln \left( \frac{n+1}{n} \right) = \ln \left( 1 + \frac{1}{n} \right) \). By l'Hôpital's rule,

\[
\lim_{n \to \infty} \left( \frac{\ln \left( 1 + \frac{1}{n} \right)}{\frac{1}{n}} \right) = \lim_{n \to \infty} \left( \frac{\left( 1 + \frac{1}{n} \right)^{1-1} \left( \frac{1}{n} \right)^2}{\left( \frac{1}{n} \right)^2} \right) = \lim_{n \to \infty} \left( \frac{1}{1 + \frac{1}{n}} \right) = 1
\]

so this converges at a rate \( O \left( \frac{1}{n} \right) \).