Math 373 — Spring 2000

Professor Barbara Osofsky

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Answers to Homework 1 (lecture 1 – Due 1/20/00)

The odd numbered exercises have answers in the book, so you can check your work. Only a few will be answered here. Even numbered ones are answered here.

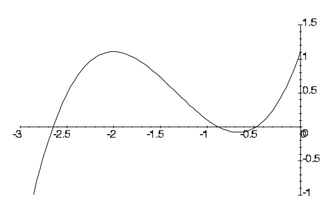
Exercise 1 (Page 14: 2) *Find intervals containing solutions to the following equations:*

(a) $x - 3^{-x} = 0$.
$[0, 1]$ will work, since $0 - 3^{-0} = -1 < 0$ and $1 - 3^{-1} = \frac{2}{3} > 0$.

(b) $4x^2 - e^x = 0$.
Again $[0, 1]$ will do since $4 \cdot 0^2 - e^0 = -1 < 0$ and $4 \cdot 1 - e^1 = 4 - e > 0$.

(c) $x^3 - 2x^2 - 4x + 3 = 0$.
When x is very small (large in absolute value but negative) the cubic is negative. When x is very large, the cubic is positive. Checking a few values, we see that for $f(x) = x^3 - 2x^2 - 4x + 3$, $f(-2) = -5$, $f(-1) = 4$, $f(0) = 3$, $f(1) = -2$, $f(2) = -5$, $f(3) = 0$. There are at most 3 roots, one of which is 3. There must be a root in $[-2, -1]$ and another in $[0, 1]$.

(d) $x^3 + 4.001x^2 + 4.002x + 1.101 = 0$.
As above, there is at least one root. For $g(x) = x^3 + 4.001x^2 + 4.002x + 1.101$ we see that $g(-3) = -1.896$ and $g(-2) = 1.101$ so there is a root in $[-3, -2]$. It is not easy to tell if there are other real roots, so we employ a graphing utility (or perhaps a root finder, but in this case I will graph):



showing that the function should be negative again at $-.5$ so we compute $g\left(-\frac{1}{2}\right) = -.02475$ so there is a root in $\left[-2, -\frac{1}{2}\right]$ and another in $\left[-\frac{1}{2}, 0\right]$ since $g(0) = 1.101$.

Exercise 2 (Page 14: 5) *Use the Intermediate Value Theorem and Rolle's Theorem to show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis precisely once, regardless of the value of the constant k.*

When x is large in absolute value but negative, the x^3 term dominates in f and the function is negative. When x is large in absolute value but positive, again the x^3 term dominates and f is positive. Thus by the intermediate value theorem f must have at least one zero. $f'(x) = 3x^2 + 2$ is always positive. If f had two zeros, say $p < q$, then Rolle's theorem would give a value c of x with $p < c < q$ such that $f'(c) = 0$. There cannot be such a c since $f'(x) > 0$ for all x. Hence there is precisely one zero.

Exercise 3 (Page 14: 6) *Suppose that $f \in C[a, b]$ and f' exists on (a, b). Show that if $f'(x) \neq 0$ for all x in (a, b), then there can exist at most one number p in $[a, b]$ with $f(p) = 0$.*

Assume there are two distinct numbers, p and q, with $a \leq p < q \leq b$ such that $f(p) = f(q) = 0$. By Rolle's theorem, there would be a $c \in (p, q) \subseteq (a, b)$ such that $f'(c) = 0$. But this contradicts our hypothesis, so no such p and q can exist.

Exercise 4 (Page 14: 12) *Let $f(x) = 2x\cos(2x) - (x-2)^2$ and $x_0 = 0$.*

(a) Find the third Taylor polynomial $P_{3,f}(x)$ and use it to approximate $f(0.4)$.

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	$f^{(n)}(x_0)/n!$	$f^{(n)}(x_0)/n!\,(x-x_0)^n$
0	$2x\cos 2x - (x-2)^2$	-4	-4	-4
1	$2\cos 2x - 4x\sin 2x - 2x + 4$	6	6	$6x$
2	$-8\sin 2x - 8x\cos 2x - 2$	-2	-1	$-x^2$
3	$-24\cos 2x + 16x\sin 2x$	-24	-4	$-4x^3$
4	$64\sin 2x + 32x\cos 2x$			

so from the table $P_{3,f}(x) = -4x^3 - x^2 + 6x - 4$.
Then $P_{3,f}(0.4) = (((-4(0.4) - 1)(0.4) + 6)(0.4) - 4) = -2.016$

(b) Use the error formula in Taylor's theorem to find an upper bound for the error $|f(0.4) - P_{3,f}(0.4)|$. Compute approximately the absolute error.

$$f(x) = P_{3,f}(x) + \frac{f^{(4)}(c)}{4!}(0.4)^4$$

for some c between 0 and x. Now $\left|f^{(4)}(c)\right| \leq 64 + 32(0.4)(1) = 76.8$ so $\left|\frac{f^{(4)}(c)}{4!}\right|(0.4)^4 \leq \frac{76.8}{24}(0.4)^4 = .08192$.
$|f(0.4) - P_{3,f}(0.4)| = |-2.002634633 + 2.016| = .013365367$. In practice one should not give so many digits. The absolute error is at most 1.34×10^{-2}.

(c) Find the 4^{th} Taylor polynomial $P_{4,f}$ and use it to approximate $f(.04)$.
$P_{4,f}(x) = P_{3,f}(x) + \frac{f^{(4)}(0)}{4!} = -4x^3 - x^2 + 6x - 4 + 0 = P_{3,f}(x)$ so the approximation is the same as above, namely -2.016.

(d) Use the error formula in Taylor's theorem to find an upper bound for the error $|f(0.4) - P_{4,f}(0.4)|$. Compute approximately the absolute error.
$f^{(5)}(x) = 160\cos 2x - 64x\sin 2x$ so $\left|f^{(5)}(x)\right| \leq 160 + 64 = 224$. More careful analysis might lower this bound, but we will use it. Then $\left|\frac{f^{(5)}(c)}{5!}\right|(0.4)^5 \leq \frac{224}{5!}(0.4)^5 = 1.911466667 \times 10^{-2}$. We have already computed approximately the absolute error.

Exercise 5 (Page 27: 10) *The number* e *can be defined by* $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ *where* $n! = n\,(n-1)\cdots 2\cdot 1$. *Compute approximately the absolute error and relative error in the following approximations of* e*:*

(a) $\sum_{n=0}^{5} \frac{1}{n!} = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120} = \frac{163}{60}$. The absolute error is $\left|e - \frac{163}{60}\right| \approx .00\,16151\,61$. The relative error is $\frac{\left|e-\frac{163}{60}\right|}{e} \approx \frac{.00\,16151\,61}{2.\,71828\,1828} \approx 5.\,9418 \times 10^{-4}$

(b) $\sum_{n=0}^{10} \frac{1}{n!} = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\frac{1}{5040}+\frac{1}{40320}+\frac{1}{362880}+\frac{1}{3628800} = \frac{9864101}{3628800} \approx 2.\,71828\,1801$. The absolute error is $\left|e - \frac{9864101}{3628800}\right| \approx 2.\,7 \times 10^{-8}$ (note the loss of significant digits) and the relative error is $\frac{\left|e-\frac{9864101}{3628800}\right|}{e} \approx 9.\,93 \times 10^{-9}$.