Math 373 — Spring 2000
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Answers to Homework 1 (lecture 1 — Due 1/20/00)
The odd numbered exercises have answers in the book, so you can check your work. Only a few will be answered here. Even numbered ones are answered here.

Exercise 1 (Page 14: 2) Find intervals containing solutions to the following equations:

(a) \( x - 3^{-x} = 0. \)

[0, 1] will work, since \( 0 - 3^{0} = -1 < 0 \) and \( 1 - 3^{-1} = \frac{2}{3} > 0. \)

(b) \( 4x^2 - e^x = 0. \)

Again [0, 1] will do since \( 4 \cdot 0^2 - e^0 = -1 < 0 \) and \( 4 \cdot 1 - e^1 = 4 - e > 0. \)

(c) \( x^3 - 2x^2 - 4x + 3 = 0. \)

When \( x \) is very small (large in absolute value but negative) the cubic is negative. When \( x \) is very large, the cubic is positive. Checking a few values, we see that for \( f(x) = x^3 - 2x^2 - 4x + 3, \) \( f(-2) = -5, f(-1) = 4, f(0) = 3, f(1) = -2, f(2) = -5, f(3) = 0. \) There are at most 3 roots, one of which is 3. There must be a root in \( [-2, -1] \) and another in \( [0, 1]. \)

(d) \( x^3 + 4.001x^2 + 4.002x + 1.101 = 0. \)

As above, there is at least one root. For \( g(x) = x^3 + 4.001x^2 + 4.002x + 1.101 \) we see that \( g(-3) = -1.896 \) and \( g(-2) = 1.101 \) so there is a root in \([-3, -2]\). It is not easy to tell if there are other real roots, so we employ a graphing utility (or perhaps a root finder, but in this case I will graph):

![Graph of a function]

showing that the function should be negative again at \(-5\) so we compute \( g\left(-\frac{1}{2}\right) = -0.2475 \) so there is a root in \([-2, -\frac{1}{2}]\) and another in \([-\frac{1}{2}, 0]\) since \( g(0) = 1.101. \)
Exercise 2 (Page 14: 5) Use the Intermediate Value Theorem and Rolle’s Theorem to show that the graph of \( f(x) = x^3 + 2x + k \) crosses the x-axis precisely once, regardless of the value of the constant \( k \).

When \( x \) is large in absolute value but negative, the \( x^3 \) term dominates in \( f \) and the function is negative. When \( x \) is large in absolute value but positive, again the \( x^3 \) term dominates and \( f \) is positive. Thus by the intermediate value theorem \( f \) must have at least one zero. \( f'(x) = 3x^2 + 2 \) is always positive. If \( f \) had two zeros, say \( p < q \), then Rolle’s theorem would give a value \( c \) of \( x \) with \( p < c < q \) such that \( f'(c) = 0 \). There cannot be such a \( c \) since \( f'(x) > 0 \) for all \( x \). Hence there is precisely one zero.

Exercise 3 (Page 14: 6) Suppose that \( f \in C[a, b] \) and \( f' \) exists on \((a, b)\). Show that if \( f'(x) \neq 0 \) for all \( x \) in \((a, b)\), then there can exist at most one number \( p \in [a, b] \) with \( f(p) = 0 \).

Assume there are two distinct numbers, \( p \) and \( q \), with \( a \leq p < q \leq b \) such that \( f(p) = f(q) = 0 \). By Rolle’s theorem, there would be a \( c \in (p, q) \subseteq (a, b) \) such that \( f'(c) = 0 \). But this contradicts our hypothesis, so no such \( p \) and \( q \) can exist.

Exercise 4 (Page 14: 12) Let \( f(x) = 2x \cos(2x) - (x - 2)^2 \) and \( x_0 = 0 \).

(a) Find the third Taylor polynomial \( P_{3, f}(x) \) and use it to approximate \( f(0.4) \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(x_0) )</th>
<th>( f^{(n)}(x_0)/n! )</th>
<th>( f^{(n)}(x_0)/n! (x - x_0)^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 2x \cos 2x - (x - 2)^2 )</td>
<td>-4</td>
<td>-4</td>
<td>-4</td>
</tr>
<tr>
<td>1</td>
<td>( 2 \cos 2x - 4x \sin 2x + 2x + 4 )</td>
<td>6</td>
<td>6</td>
<td>6x</td>
</tr>
<tr>
<td>2</td>
<td>( -8 \sin 2x - 8x \cos 2x - 2 )</td>
<td>-2</td>
<td>-1</td>
<td>-x^2</td>
</tr>
<tr>
<td>3</td>
<td>( -24 \cos 2x + 16x \sin 2x )</td>
<td>-24</td>
<td>-4</td>
<td>-4x^3</td>
</tr>
<tr>
<td>4</td>
<td>( 64 \sin 2x + 32x \cos 2x )</td>
<td>so from the table ( P_{3, f}(x) = -4x^3 - x^2 + 6x - 4 ).</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then \( P_{3, f}(0.4) = (((-4)(0.4) - 1)(0.4) + 6)(0.4) - 4 = -2.016 \)

(b) Use the error formula in Taylor’s theorem to find an upper bound for the error \( |f(0.4) - P_{3, f}(0.4)| \). Compute approximately the absolute error.

\[
f(x) = P_{3, f}(x) + \frac{f^{(4)}(c)}{4!} (0.4)^4
\]

for some \( c \) between 0 and \( x \). Now \( \left| f^{(4)}(c) \right| \leq 64 + 32 (0.4) (1) = 76.8 \) so \( \left| \frac{f^{(4)}(c)}{4!} \right| (0.4)^4 \leq \frac{76.8}{24} (0.4)^4 = 0.8192 \).

\( |f(0.4) - P_{3, f}(0.4)| = |-2.002634633 + 2.016| = 0.13365367 \). In practice one should not give so many digits. The absolute error is at most \( 1.34 \times 10^{-2} \).

(c) Find the 4th Taylor polynomial \( P_{4, f} \) and use it to approximate \( f(0.4) \).

\( P_{4, f}(x) = P_{3, f}(x) + \frac{f^{(4)}(0)}{4!} = -4x^3 - x^2 + 6x - 4 + 0 = P_{3, f}(x) \) so the approximation is the same as above, namely \(-2.016\).

(d) Use the error formula in Taylor’s theorem to find an upper bound for the error \( |f(0.4) - P_{4, f}(0.4)| \). Compute approximately the absolute error.

\( f^{(4)}(x) = 160 \cos 2x - 64x \sin 2x \) so \( \left| f^{(4)}(0) \right| \leq 160 + 64 = 224 \). More careful analysis might lower this bound, but we will use it. Then \( \left| \frac{f^{(4)}(0)}{4!} \right| (0.4)^5 \leq \frac{224}{24} (0.4)^5 = 1.911466667 \times 10^{-2} \). We have already computed approximately the absolute error.
Exercise 5 (Page 27: 10) The number $e$ can be defined by $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ where $n! = n(n-1) \cdots 2$.

1. Compute approximately the absolute error and relative error in the following approximations of $e$:

   (a) $\sum_{n=0}^{5} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60}$. The absolute error is $|e - \frac{163}{60}| \approx 0.01615161$. The relative error is $\frac{|e - \frac{163}{60}|}{e} \approx \frac{0.01615161}{2.718281828} \approx 5.9418 \times 10^{-4}$.

   (b) $\sum_{n=0}^{10} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320} + \frac{1}{362880} + \frac{1}{3628800} = \frac{9864101}{3028800} \approx 2.718281801$. The absolute error is $|e - \frac{9864101}{3028800}| \approx 2.7 \times 10^{-8}$ (note the loss of significant digits) and the relative error is $\frac{|e - \frac{9864101}{3028800}|}{e} \approx 9.93 \times 10^{-9}$.  
