Math 373 **HOURLY 2** **April 6, 2000**

ANSWERS Name ____________________

1. Your boss wants to know the solution of the initial value problem

$$y' = 2\frac{y}{x} + 1 \qquad y(1) = -1$$

on the interval $[1, 2]$.

(a) Show that there is a unique solution to this initial value problem on the interval $[1, 2]$.

You should use the main theorem on existence and uniqueness which says that if $f(x, y)$ satisfies a Lipschitz condition on $\{(x, y) : a \le x \le b,\ -\infty < y < \infty\}$ then there is a unique solution to the IVP $y' = f(x, y)$, $y(a) = \alpha$ on the interval $[a, b]$, and the problem is well posed. For this example, it is easy to show directly that the function does indeed satisfy a Lipschitz condition on $\{(x, y) : 1 \le x \le 2,\ -\infty < y < \infty\}$.
Fix x, and let y and z be any two real numbers. Then $\left|\left(2\frac{y}{x} + 1\right) - \left(2\frac{z}{x} + 1\right)\right| = \left|\frac{2}{x}(y - z)\right| = \left|\frac{2}{x}\right| |y - z|$. Since $x \in [1, 2]$ and $\frac{2}{x}$ is a decreasing function, $\left|\frac{2}{x}\right| \le 2$ on $[1, 2]$ so $\left|\left(2\frac{y}{x} + 1\right) - \left(2\frac{z}{x} + 1\right)\right| \le 2|y - z|$ on the infinite rectangle and $2\frac{y}{x} + 1$ does satisfy a Lipschitz condition with Lipschitz constant 2 (or, if you are really very fussy, $2 + \varepsilon$ for any positive real number ε to make that inequality strict.

(b) Use Euler's method with $h = 0.5$ to get a (not very accurate) numerical solution to this IVP on $[1, 2]$.

The Euler's method difference equation is

$$w_{i+1} = w_i + h \cdot f(x_i, w_i).$$

We then have the numerical solution

i	x_i	w_i	$f(x_i, w_i)$	$w_{i+1} = w_i + h \cdot f(x_i, w_i)$
0	1.0	-1.0	$2\frac{(-1)}{1} + 1 = -1$	$-1 + .5(-1) = -1.5$
1	1.5	-1.5	$2\frac{(-1.5)}{1.5} + 1 = -1$	$-1.5 + .5(-1) = -2.0$
2	2.0	-2.0		

2. You are making two approximations to $\int_0^{0.2} F(x)\,dx$ where $F(0.1) = 5$. The first approximation is using the Trapezoid rule with two subdivisions. The second approximation is using simple Simpson's rule. This second approximation gives the numerical approximation $\int_0^{.2} F(x)\,dx \approx .9532$. What value does the first (compound Trapezoid with $n = 2$) approximation give?

Using the trapezoid rule we get an approximation I_T for the integral, and using Simpson we get an approximation I_S where

$$\begin{aligned} I_T &= \frac{0.1}{2.}\left(F(0.0) + 2\cdot F(0.1) + F(0.2)\right) = (0.05)\left(F(0.0) + 2\cdot 5 + F(0.2)\right) \\ I_S &= \frac{0.2}{6}\left(F(0.0) + 4\cdot F(0.1) + F(0.2)\right) = \frac{1}{30}\left(F(0.0) + 4\cdot 5 + F(0.2)\right) = .9532\ . \end{aligned}$$

Then multiplying the second equation by 30 and solving for $F(0.0) + F(0.2)$ gives

$$F(0.0) + F(0.2) = 30(.9532) - 20 = 8.596$$

and substituting in the first gives $I_T = (0.05)(8.596 + 10) = .9298$.

3. You wish to approximate the definite integral

$$\int_1^3 \frac{1}{(x+1)^2}\,dx$$

using composite Simpson's rule.

(a) If the interval $[1, 3]$ is divided up into 20 subintervals, approximate the truncation error in your computed value. Do not compute the actual approximation to the integral—just approximate the error.

The composite Simpson's rule error term is

$$E(f) = -\frac{b-a}{180}\,h^4 f^{(4)}(\mu)$$

where the interval of integration, $[a, b]$ is divided into n equal subintervals for the even positive integer n, $h = \frac{b-a}{n}$, and $\mu \in (a, b)$. See page 201 of the text. Here the function $f(x) = \frac{1}{(x+1)^2}$ has $f'(x) = -\frac{2}{(x+1)^3}$, $f''(x) = \frac{6}{(x+1)^4}$, $f'''(x) = -\frac{24}{(x+1)^5}$, $f^{(4)}(x) = \frac{120}{(x+1)^6}$. Increasing the denominator decreases the value of the fraction, so the maximum possible value of $f^{(4)}(x)$ on $[1, 3]$ occurs when $x = 1$, and $f^{(4)}(1) = \frac{15}{8}$. Then

$$|E(f)| \le \frac{2}{180}\left(\frac{2}{20}\right)^4 \frac{15}{8} = 2.08333 \times 10^{-6}$$

(b) Determine the number of subintervals, n, required to approximate this integral to within 5×10^{-6} using composite Simpson's rule.

From the first part of this question, $|E(f)| \le \frac{2}{180}(h)^4 \frac{15}{8}$ and this is at most 5×10^{-6} if and only if

$$\begin{aligned} \frac{2}{180}(h)^4 \frac{15}{8} &= \frac{1}{48}(h)^4 \le 5 \times 10^{-6} \\ 4\ln(h) - \ln(48) &\le \ln\left(5 \times 10^{-6}\right) \end{aligned}$$

$$\begin{aligned}
\ln(h) &\le \frac{\ln(5\times10^{-6})+\ln(48)}{4} = -2.08371\\
\frac{2}{n} &= h \le \exp(-2.08371) = .12446\\
n &\ge \frac{2}{.12446} = 16.06941
\end{aligned}$$

Since n is an even integer, this means that to be sure you are within 5×10^{-6} you should take at least 18 subintervals.

4. You are given the following data about a function $g(x)$:

x	1	1.1	1.2
$g(x)$	1.55740	1.96475	2.57215

(a) Derive a formula, together with error estimate, that will enable you to approximate $f''(x_0)$ for any function $f(x)$ with a convergent Taylor series expansion in an interval about the x-value x_0 provided you have the values $f(x_0)$, $f(x_0+h)$, and $f(x_0+2h)$ (and no other values).

We have :
$g(x_0) = g(x_0)$
$g(x_0+h) = g(x_0) + g'(x_0)h + g''(x_0)\frac{h^2}{2} + g'''(\xi_1)\frac{h^3}{6}$
$g(x_0+2h) = g(x_0) + 2g'(x_0)h + 4g''(x_0)\frac{h^2}{2} + 8g'''(\xi_2)\frac{h^3}{6}$.
From these we get
$\left[g(x_0+h) - g(x_0) = g'(x_0)h + g''(x_0)\frac{h^2}{2} + g'''(\xi_1)\frac{h^3}{6}\right]$
$\left[g(x_0+2h) - g(x_0) = 2g'(x_0)h + 4g''(x_0)\frac{h^2}{2} + 8g'''(\xi_2)\frac{h^3}{6}\right]$
Multiply the first of these equations by 2 and subtract from the second to get
$[g(x_0+2h)-g(x_0)] - 2[g(x_0+h)-g(x_0)]$
$\quad = 2g'(x_0)h + 4g''(x_0)\frac{h^2}{2} + 8g'''(\xi_2)\frac{h^3}{6} - 2\left(g'(x_0)h + g''(x_0)\frac{h^2}{2} + g'''(\xi_1)\frac{h^3}{6}\right)$
$\quad = g''(x_0)h^2 + \left(\frac{4}{3}g'''(\xi_2) - \frac{1}{3}g'''(\xi_1)\right)h^3$
You may leave that error term as is, or you may reduce, as in the book, to the third derivative evaluated at a single point. I also accepted an order of convergence term $O(h^3)$. Then

$$\begin{aligned}
g(x_0+2h) + g(x_0) - 2g(x_0+h) &= g''(x_0)\left(2h^2-h^2\right) + g'''(\xi)h^3\\
g''(x_0) &= \frac{g(x_0+2h)+g(x_0)-2g(x_0+h)}{h^2} + g'''(\xi)h.
\end{aligned}$$

(b) Use this formula to approximate $g''(x_i)$ for the above function $g(x)$ and x_i one of 1, 1.1, 1.2.

$g''(1) \approx \frac{1.55740+2.57215-2*1.96475}{(.1)^2} = 20.005$

(c) The book gives a formula approximating the second derivative of a function f as

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right] - \frac{h^2}{24} f^{(4)}(\xi)$$

for some ξ with $x_0 - h < \xi < x_0 + h$.
At what point (x value) will this formula enable you to approximate $g''(x)$ for g the function in the preamble to this question (question 4)?

This formula uses values of the function both to the right and to the left of the point where the derivative is being evaluated, so it can be used to approximate $g''(x)$ for $x = 1.1$.

(d) Which of the two formulas, the one derived in 4.a or the one derived in 4.c, would you expect to have the smaller truncation error and briefly say why? (Note: it is possible to give a good answer to this even if you do not do 4.a.)

I would expect the symmetric one to be better for two reasons. By taking points on both sides of the evaluation point, your data as a whole is closer to the point of evaluation and intuitively that should be better. More specifically, the formula in 4.c has a higher order (in h) error term so unless the derivatives are somewhat funny, one would expect it to have smaller truncation error. Notice that these two formulas are not computing the same number, but approximating the second derivative at two different points.

5. The simple midpoint rule says that, for nice functions $f(x)$, there are constants K_i with

$$\int_{x_0-h}^{x_0+h} f(x)\, dx = 2hf(x_0) + K_1h^3 + K_2h^5 + K_3h^7 + \cdots$$

so that there is a composite midpoint rule

$$\int_a^b f(x)\,dx = 2h\left(\sum_{j=0}^{n-1} f(a + (2j+1)h)\right) + K_1'h^2 + K_2'h^4 + K_3'h^6 + \cdots$$

where $h = \frac{b-a}{2n}$ (that is, the interval $[a, b]$ is divided up into $2n$ equal subintervals and the midpoint rule is applied on each interval $[a + jh\,,\, a + (j+2)h]$ for $j = 0, 1, \cdots, n-2$.

(a) You are given the data about a function G

x	.25	.50	.75	1	1.25	1.50	1.75
$G(x)$	0.15749	0.39685	0.68142	1.00000	1.34652	1.71707	2.10887

Starting with $x_0 = 1$ and $n = 1$ and the initial approximation $N(h)$ given by the composite midpoint rule

$$\begin{aligned} N(h) &= 2h\left(\sum_{j=0}^{n-1} f(a + (2j+1)h)\right) \\ &= 2h\left[f(a+h) + f(a+3h) + f(a+5h) + \cdots + f(a + (2n-1)h)\right] \end{aligned}$$

use Richardson's extrapolation as far as you can with the given data to approximate $\int_0^2 G(x)\,dx$.

$$
\begin{aligned}
N_1(1) &= 2 \cdot 1 \cdot G(1) = 2.0 \\
N_1(.5) &= 2\tfrac{1}{2}\left(G(.5) + G(1.5)\right) = 0.39685 + 1.71707 = 2.11392 \\
N_1(.25) &= 2\tfrac{1}{4}\left(0.15749 + 0.68142 + 1.34652 + 2.10887\right) = 2.14715 \\
N_2(1) &= \tfrac{4(2.11392)-2.0}{3} = 2.15189 \\
N_2(.5) &= \tfrac{4(2.14715)-2.11392}{3} = 2.15822 \\
N_3(1) &= \tfrac{16(2.15822)-2.15189}{15} = 2.158642
\end{aligned}
$$

(b) Why might one use this numerical integration method rather than Romberg integration?

It saves a little computation but note that we have used every value given at some point so not much is saved. To use standard Romberg, we would need two more functional evaluations at the endpoints, and that is the main computational difference. However, if there is any problem at the endpoints, such as an improper integral or values not measured as here, this does not need those endpoint functional values whereas Romberg does.

6. You are using adaptive Simpson's rule to compute the integral of some function $H(x)$ from 1 to 2. You have used simple Simpson's rule to compute the following approximations:

$$I = \int_1^2 H(x)\,dx \approx .5117550$$

$$I_L = \int_1^{1.5} H(x)\,dx \approx .1324504 \qquad I_R = \int_{1.5}^{2} H(x)\,dx \approx .3800911$$

$$I_{LL} = \int_1^{1.25} H(x)\,dx \approx .03428033 \qquad I_{LR} = \int_{1.25}^{1.5} H(x)\,dx \approx .09837382$$

$$I_{RL} = \int_{1.5}^{1.75} H(x)\,dx \approx .1599144 \qquad I_{RR} = \int_{1.75}^{2} H(x)\,dx \approx .2201797$$

$$I_{LLL} = \int_1^{1.125} H(x)\,dx \approx .008872308 \qquad I_{LLR} = \int_{1.125}^{1.25} H(x)\,dx \approx .02546074$$

$$I_{LRL} = \int_{1.25}^{1.375} H(x)\,dx \approx .04138843 \qquad I_{LRR} = \int_{1.375}^{1.5} H(x)\,dx \approx .05698607$$

$$I_{RLL} = \int_{1.5}^{1.625} H(x)\,dx \approx .07235755 \qquad I_{RLR} = \int_{1.625}^{1.75} H(x)\,dx \approx .08755694$$

$$I_{RRL} = \int_{1.75}^{1.875} H(x)\,dx \approx .1026176 \qquad I_{RRR} = \int_{1.875}^{2} H(x)\,dx \approx .1175621$$

(a) If you wish a maximum error of at most 5×10^{-4}, what value will you return for the integral. Explain your work.

There is a factor of 15 that occurs in adaptive Simpson's based on assumptions about the 4^{th} derivative that may or may not hold. Here I will not use that factor (that is, replace it by 1 as I did in class). Nothing in the problem prevented students from using it.

$I - I_L - I_R = .5117550 - .1324504 - .3800911 = -.0007865$ which is not within my tolerance (unless one uses the factor of 15).

$I_L - I_{LL} - I_{LR} = .1324504 - .03428033 - .09837382 = -.00020375$ which has absolute value less than $\frac{5\times10^{-4}}{2}$ so this is within tolerance and we approximate $\int_1^{1.5} H(x)\,dx \approx I_{LL} + I_{LR} = .03428033 + .09837382 = .13265415$.

(This was actually a typo–the intent was to set the error at 4×10^{-4} so this failed also. For your reference, had that been done, one would next do $I_{LL} - I_{LLL} - I_{LLR} = .03428033 - .008872308 - .02546074 = -.000052718$ which has absolute value $< \frac{4\times10^{-4}}{4}$ so we would take $\int_1^{1.25} H(x)\,dx \approx .008872308 + .02546074 = .034333048$ and then look at $I_{LR} - I_{LRL} - I_{LRR} = .09837382 - .04138843 - .05698607 = -6.8 \times 10^{-7}$ which is also well within the required tolerance so we would take $\int_{1.25}^{1.5} H(x)\,dx \approx .04138843 + .05698607 = .0983745$.)

$I_R - I_{RL} - I_{RR} = .3800911 - .1599144 - .2201797 = -3.0 \times 10^{-6}$ which has absolute value within my tolerance so we approximate $\int_{1.5}^{2} H(x)\,dx \approx I_{RL} + I_{RR} = .1599144 + .2201797 = .3800941$ and the value we return for the integral is $.13265415 + .3800941 = .51274825$.

(Had the typo not occurred, the returned value would have been $.3800941 + .0983745 + .034333048 = .512801648$.

(b) If you were to use composite Simpson's rule with 8 subdivisions to approximate this integral, what value would you return.

$.008872308 + .02546074 + .04138843 + .05698607 + .07235755 + .08755694 + .1026176 + .1175621 = .512801738$.

(c) For what values of x did the composite Simpson's rule require an evaluation of $H(x)$ whereas the adaptive Simpson's rule did not.

The values of x where one did not need an evaluation of $H(x)$ in adaptive Simpson's were $x = 1.125, 1.375, 1.625, 1.875$. Without the typo they would have been just $x = 1.625, 1.875$.