Math 373: 01 Final Exam Prof. Bumby
December 20, 2000 8:00 – 11:00 PM

You are expected to have books, notes and calculators available, but computers of telephones are not to be used during the exam. You should check that you have a complete exam. There are 6 problems on 3 pages (printed single sided). All work for this exam is to be done in the blue books provided. You may keep the question paper.

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1. (35 pts.) Let
$$g(x) = 0.1 + 0.6 * \cos(2x).$$
Investigate the possibility of solving $x = g(x)$ by iterating g (as in Algorithm 2.2). If you find an interval for which the hypotheses of theorems 2.2 and 2.3 are satisfied, use it to estimate the number of iterations needed to get 10 decimal place accuracy for the fixed point. In any case, show that there is only one solution of $x = g(x)$. If convergence is uncertain (or slow), try a different rootfinding procedure. Describe your chosen procedure and apply it to give 10 decimal place accuracy. Can you now find an interval satisfying the hypotheses of theorems 2.2 and 2.3?

solution

2. (35 pts.) Let $y \geq 1$ be given as a function of $x \geq 0$ by
$$y^3 - y = x.$$
(The restriction on x and y guarantees that a unique root of the cubic is selected for all admissible values of x). The inverse function theorem allows us to tabulate some values of this function and its derivative at irregularly spaced values of x (chosen to give nice values of y). Here is such a table.

x	y	dy/dx
0.000	1.0	1/2.00
0.231	1.1	1/2.63
0.528	1.2	1/3.32
0.897	1.3	1/4.07
1.344	1.4	1/4.88
1.875	1.5	1/5.75
2.496	1.6	1/6.68
3.213	1.7	1/7.67

Use this to construct a divided difference table to give an appropriate interpolating polynomial of degree 2 for approximating the value of y when $x = 1$. What value do you get for y? To test the accuracy, compute $y^3 - y$ What does this say about the accuracy of the value of y? Now, use values of y and dy/dx at appropriate values of x to construct a divided difference table to give a Hermite cubic to approximate the value of y when $x = 2$. Test the accuracy of this as before.

solution

3. (30 pts.) By using the Taylor series for $\cos(x)$, one can prove that

$$\frac{1-\cos x}{x^2} = \frac{1}{2} - \frac{x^2}{24} + \frac{x^4}{720} - \frac{x^6}{40320}\cos\xi$$

for some ξ between 0 and x. For which x will this formula compute the function accurately to within 10^{-10}? Apply the formula to evaluate the function at $x = .1$, $x = .05$, $x = .01$, $x = .005$ and $x = .001$ to the accuracy of your calculator. Compare the answers to the value obtained by subtracting the value given by the `cos` key on your calculator from 1 and dividing by x^2. They should be (slightly) different. Explain the difference.

solution

4. (30 pts.) Here is a portion of a table of a function.

x	y
0.3	1.531475
0.4	1.402252
0.5	1.284457
0.6	1.177036
0.7	1.079035

Approximate the derivative of this function at each of the given values of x using the best *three-point* numerical differentiation rule for each point. **Show the formula being used as well as the answer.** Also use the *five-point* rule of equation (4.6) of the textbook to obtain what should be a much better estimate of the derivative at 0.5. Assuming that this is a much better estimate, how small would you expect the **truncation** error to be if you used the three point formula based on values of the function with x increments of 0.01 instead of 0.1? If the function was still tabulated to 6 decimal places, what would the **round-off** error be?

solution

5. (35 pts.) Devise a plan to find

$$\int_{1.5}^{2} \frac{e^x}{x}\,dx$$

to 8 decimal places. Begin with the simplest form of the trapezoidal rule and repeatedly half the step size in the **composite** trapezoidal rule. Use this as a basis of a **Romberg method** to develop higher order rules. Even if the Romberg method does not yield a quick answer, it should allow you to estimate the number of times you must halve the step size to get an acceptable answer with the composite trapezoidal rule and with the composite Simpson's rule. Give the step sizes needed for these rules. If excessive computation would be required to give a suitable value of the integral, describe how the problem would be done on a computer. (If your calculator has a built-in Simpson's rule that you have used and you have determined that it will give a suitable answer, use it and describe how you entered the data for this method and the answer obtained. Otherwise, describe your choice of the best method for obtaining an answer and any programming considerations.)

solution

6. (35 pts.) Consider the differential equation

$$\frac{dy}{dt} = \frac{y}{t} - \left(\frac{y}{t}\right)^2$$

with $y(1) = 1$. Select a **second-order method** (either a Taylor method or one of the three second-order Runge-Kutta methods described on pages 279-280 of the textbook) and compute **two steps** with $h = 0.1$. Then perform a single step of the fourth-order Runge-Kutta method with a step size of $h = 0.2$. (Since the solution of the equation is known, we have that the value of y when $t = 1.2$ should be 1.01495231404. How accurate are your results?) Outline a program for solving this equation for $1 \le t \le 2$ to six decimal place accuracy. Include a test that will check the accuracy of the answer that does not depend on having an exact solution to the equation.

solution

End of exam.