

Math 373: 01 — Fall 2000  
MW8 SC-205  
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Mathematics at Rutgers is making greater use of the *World Wide Web*. Paper handouts like this will serve mainly as guides to other information.

The mathematics department home page at

<http://www.math.rutgers.edu/>

contains links to “Faculty” and “Course Materials” which lead to lists from which you can locate individual instructors or courses. The first time you look for a page for one of your courses, you should take the scenic route by selecting links starting from the department home page.

For this course, there is a “373f00” Web Page that will be updated regularly and will be the source for homework assignments. Material that needs to display mathematical notation will be presented as separate documents in PDF format. In particular, there will be lists of supplementary problems that will allow the course to move beyond the confines of the textbook exercises on short notice. Some of these will be lecture topics while others are to be done for homework. The role of each exercise will be announced in class and posted on the Web when the relevant section is treated in class. The rest of this handout will be devoted to the first such list.

Supplementary Problem List #1

1. (for Sect. 2.1, using a calculator) Consider

$$f(x) = x \tan x$$

(a) Show that  $f(x)$  an *increasing function* for  $0 \leq x < \pi/2$  with

$$f(0) = 0 \quad \text{and} \quad \lim_{x \rightarrow \pi/2} = +\infty.$$

(b) Find  $f(1)$  to six decimal places (i.e., with six digits after the decimal point). You should find that it is greater than 1, so  $f(x) = 1$  will have a solution between 0 and 1. (If you get a very small value for  $f(1)$ , your calculator is probably in *degree mode*. Change to *radian mode*, and keep it there. Calculators also have a *grad mode* that is rarely used. In that mode, you would get a value of approximately 1.57 when you try to compute  $f(1)$ . The use of trigonometric functions in mathematics requires angles to be measured in radians, so the use of any other mode on your calculator is a mistake. The only correct value of  $f(1)$  is between 1.55 and 1.56.)

(c) Use the **bisection method** on the interval  $[0, 1]$  to get an interval of length  $2^{-10}$  containing the root of  $f(x) = 1$  between 0 and 1. Save this answer — it will be used in a later problem.

... more on other side

2. (for Sect. 2.2, using either a programmable calculator or a computer) Consider

$$g_a(x) = \cos ax$$

for some real number  $a$  which will be fixed in each part of this problem, although different values of  $a$  will be used in different parts.

(a) Clearly, for all  $a$ , the function  $g_a(x) \in [-1, 1]$  for all  $x$ , so  $g_a$  maps  $[-1, 1]$  into itself. The other part of the hypothesis of **Theorem 2.3** (page 61), that  $|g'_a(x)| \leq k < 1$  can also be verified for the interval  $[-1, 1]$  for  $a = 1$ . Indeed, you know enough about  $g_a$  and its derivative that you can find

$$k_0 = \text{lub} \{ |g'_a(x)| : -1 \leq x \leq 1 \},$$

which is the smallest value of  $k$  that works for this interval. Program the iteration of  $g_a$  on your calculator in in your favorite programming language and verify that that the iteration converges. Report a **sample** of the iteration to get a rough idea of the rate of convergence.

(b) Show that both hypotheses of Theorem 2.3 also hold for the interval  $[0, 1]$  and for  $[.5, .9]$  with  $k = k_0$ . Here, you can need to show that the  $g_a$  maps the interval into itself since you *know* that  $|g'_a(x)| \leq k_0$  for all  $x$  in the interval. (You do know it, don't you?) A smaller value of  $k$  will work for  $[.5, .9]$ . Find such a  $k$ .

(c) Now try  $a = 1.1$ . You should find that you can still verify both hypotheses of Theorem 2.3 for each of the intervals considered above. However, a larger value of  $k$  needs to be used. Modify your program from (a) to apply here and verify that that the iteration converges. Report a **sample** of the iteration to get a rough idea of the rate of convergence.

(d) Since the conclusion of Theorem 2.3 gives a fixed point of the function  $g$ , the only intervals for which the hypotheses can be satisfied are those which contain a fixed point. If  $\xi$  is a fixed point of  $g$ , then  $|g'_a(\xi)| \leq k$ . For  $a = 2$ , a graph reveals that there is only one fixed point  $\xi$  and  $g'_a(\xi) < -1$ . Thus, iteration of  $g_a$  will not converge. Modify your program to calculate a few values. You should see an example of *chaotic* behavior. The significance of the first exercise is that it can be used to show that iteration converges to a fixed point if  $a < 1.319156505 \dots$ . Slightly larger values of  $a$  lead to iterations that remain well-behaved even if they don't converge, but much larger values are chaotic.

(e) When  $a = 3$ , there will again be an interval on which the fixed point theorem applies. First do some iterations with different starting values. Then graph  $y = g_a(x)$  to help identify an interval for which the hypothesis of the fixed point theorem. The details need not be supplied in this part.

3. (for Sect. 2.3) Apply Newton's method to the equation  $x_n = c$  to find a function  $h_n(x)$  such that iterating  $h_n$  will converge to an  $n^{\text{th}}$  root of  $c$ .

(a) Consider the special case  $n = 2$ . Show that

$$h_2(x) = \frac{1}{2} \left( x + \frac{c}{x} \right).$$

Use the fact that Newton's method essentially doubles accuracy at each step to write an expression that will allow you to find 20 correct digits of the decimal expansion of  $\sqrt{10}$  starting from the 10 correct digits that your calculator gives you.

(b) Consider the special case  $n = -2$ . Show that  $h_{-2}(x)$  can be evaluated without performing any division. Use the fact that  $\sqrt{10}$  is the solution of  $x^{-2} = 0.1$  to write another formula that can also be expected to give 20 correct decimal digits starting from the calculator value of  $\sqrt{10}$ .

(c) Without using a package like *Maple* that includes high precision arithmetic, use the result of (a) or (b) to compute a 20 digit approximation to  $\sqrt{10}$ . If necessary, use pencil-and-paper methods, but you should be able to find a way of programming such computations using blocks of 4 or 5 digits (you need to be sure that the product of two such blocks can be found exactly on your machine).