Math 373: 01 — Fall 2000
MW8   SC–205
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Handout 2: Pi and the AGM

The title of this handout is borrowed from reference [1]. This is the standard introduction to the techniques used for extremely high precision computation. The algorithms described here are also featured in the classical text [3]. A strikingly simple proof of the formula for the limit of this iteration, with a discussion of its deeper significance appears in [2].

This material extends the discussion of *order of convergence* in Section 2.4. Problems S4 and S5 supplement the problems from that section of the text. There is also a problem S6 that supplements Section 2.5. Section 2.6 will be skipped.

Calculation of π by Archimedes. Archimedes proved that

$$\frac{223}{71} < \pi < \frac{22}{7}$$

by calculating the perimeters of regular polygons of 96 sides inscribed and circumscribed about a unit circle. (An easy argument shows that the inscribed polygon has perimeter less than 2π and the circumscribed polygon has perimeter greater than 2π). The reason for using 96 sides was that he started with a hexagon and applied a formula that related the perimeters of the polygons with n sides to those with $2n$ sides. You can do much better with your calculator (although you don't have to since the calculator has the value of π stored to the limit of its accuracy). In modern notation, the algorithm is simple. Trigonometry gives that the perimeter I_n of an inscribed polygon of n sides and the perimeter O_n of a circumscribed polygon of n sides are

$$\begin{aligned} I_n &= 2n \sin\frac{\pi}{n} = 4n \sin\frac{\pi}{2n}\cos\frac{\pi}{2n} \\ O_n &= 2n \tan\frac{\pi}{n} = \frac{4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}{\cos^2\frac{\pi}{2n} - \sin^2\frac{\pi}{2n}}. \end{aligned}$$

This shows that $\alpha_n = 1/O_n$ and $\beta_n = 1/I_n$ will be easier to work with, and

$$\begin{aligned} \frac{1}{2}(\alpha_n + \beta_n) &= \frac{\cos^2\frac{\pi}{2n}}{4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}} \\ &= \frac{\cos\frac{\pi}{2n}}{4n\sin\frac{\pi}{2n}} = \alpha_{2n} \end{aligned}$$

Then,

$$\alpha_{2n}\beta_n = \frac{1}{16n^2\sin^2\frac{\pi}{2n}} = \beta_{2n}^2.$$

The α's are increasing and the β's are decreasing. The common limit can be used to find the perimeter of the circle.

If we start with a hexagon, $I_6 = 6$ and $O_6 = 4\sqrt{3}$, and the limiting values of I_n and O_n as $n \to \infty$ is 2π. (One could also start with a square, for which $I_4 = 4\sqrt{2}$ and $O_4 = 8$, and the same limiting values would be reached.) More generally, if $0 < \alpha < \beta$ and we write $\alpha = \beta\cos\theta$, then defining $\alpha' = (\alpha + \beta)/2$ and $\beta' = \sqrt{\alpha'\beta}$, we find that

$$\alpha' = \beta'\cos(\theta/2) \text{ and } \beta'\sin(\theta/2) = (\beta\sin\theta)/2.$$

Thus, each step divides the angle θ in half while preserving $(\beta\sin\theta)/\theta$, and the limit of $\beta^{(n)}$ is the original value of $(\beta\sin\theta)/\theta$. Moreover,

$$\begin{aligned}\beta - \alpha' &= (\beta - \alpha)/2\\ \beta' - \alpha' &= \sqrt{\alpha'}\left(\sqrt{\beta} - \sqrt{\alpha'}\right)\\ &= \frac{\sqrt{\alpha'}}{\sqrt{\beta} + \sqrt{\alpha'}}(\beta - \alpha')\end{aligned}$$

which shows that $\beta - \alpha$ is reduced by a factor of slightly more than 4 at each step. If $\alpha > \beta > 0$, a similar analysis of the construction can be performed: convergence is at a comparable rate, and the limit evaluated in terms of hyperbolic functions.

Exercise S4. Use this iteration to find the bounds on π from the 96 sided polygon and show that it implies the bounds attributed to Archimedes.

The AGM of Gauss. Instead of alternately calculating arithmetic and geometric means, Gauss tried to calculate the two means in parallel. That is, start with $\alpha > \beta > 0$ and set $\alpha' = (\alpha + \beta)/2$ and $\beta' = \sqrt{\alpha\beta}$. This makes a vast difference in the rate of convergence, since

$$\begin{aligned}\alpha' - \beta' &= \frac{\left(\sqrt{\alpha} - \sqrt{\beta}\right)^2}{2}\\ &= \frac{(\alpha - \beta)^2}{2\left(\sqrt{\alpha} + \sqrt{\beta}\right)^2}\\ &< \frac{(\alpha - \beta)^2}{8\beta}\end{aligned}$$

This gives quadratic convergence. The common limit of the $\alpha^{(n)}$ and $\beta^{(n)}$ is called the *arithmetic-geometric mean* (or AGM, for short) of α and β. Unfortunately, it is more difficult to find an analytic expression for the limit than for the other iteration. Gauss had computed 20 decimal places of the AGM of 1 and $\sqrt{2}$ in 1791, and his diary entry for May 30, 1799 notes that the value of

$$\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1 - t^4}},$$

that he computed at that time seemed to be the same number. He sensed that this was important, and tried to prove it. The diary entry of December 23, 1799 records his success. Gauss did this by generalizing the identity and expanding the quantities he wanted to relate in a power series. Later proofs found an expression in terms of α and β that was equal to the corresponding expression in terms of α' and β' as in our evaluation of the limit of Archimedes' iteration.

Exercise S5. Find the AGM of 1 and $\sqrt{2}$ to the full accuracy of your calculator, and record the number of iterations required. How many did Gauss need to get 20 decimal places? How many iterations would be required for 100 decimal places? (Optionally, if you are comfortable with a multi-precision package like *Maple*, you should be able to get the value to this accuracy).

An identity. This is extracted from [2]. Some scratch work may be needed to justify all the steps. Let

$$I(a,b) = \int_0^{\pi/2} (a^2\cos^2\theta + b^2\sin^2\theta)^{-1/2} d\theta$$
$$J(a,b) = \int_0^{\pi/2} \left\{(a\cos^2\phi + b\sin^2\phi)(a\sin^2\phi + b\cos^2\phi)\right\}^{-1/2} d\phi$$

The substitution $\tan\phi = \sqrt{b/a}\tan\theta$ shows that $J(a,b) = I(a,b)$. Moreover, the expression under the radical in $J(a,b)$ can be written

$$(a\cos^2\phi + b\sin^2\phi)(a\sin^2\phi + b\cos^2\phi) = a_1^2\cos^2 2\phi + b_1^2\sin^2 2\phi$$

where $a_1 = (ab)^{1/2}$ and $b_1 = (a+b)/2$. Thus

$$\begin{aligned} I(a,b) = J(a,b) &= \int_0^{\pi/2} (a_1^2\cos^2 2\phi + b_1^2\sin^2 2\phi)^{-1/2} d\phi \\ &= \frac{1}{2}\int_0^{\pi} (a_1^2\cos^2\theta_1 + b_1^2\sin^2\theta_1)^{-1/2} d\theta_1 \qquad (\theta_1 = 2\phi) \\ &= \int_0^{\pi/2} (a_1^2\cos^2\theta_1 + b_1^2\sin^2\theta_1)^{-1/2} d\theta_1 = I(a_1,b_1)\ . \end{aligned}$$

Iterating the transformation $(a,b) \mapsto (a_1,b_1)$ gives sequences of a_k and b_k that converge to the AGM μ of a and b. Thus

$$I(a,b) = I(\mu,\mu) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\mu^2\cos^2\theta + \mu^2\sin^2\theta}} = \frac{\pi}{2\mu}$$

The changes of variables relating the integrals I and J here seem slightly more natural than those in an earlier proof of a similar identity by B. C. Carlson.

All references give additional interpretations of this identity. There are also many references to other articles on the AGM in [1].

Exercise S6. Show that Steffensen's method can do more than just accelerate convergence — it can turn a divergent process into a rapidly converging one! Apply the method with $g(x) = \cos(2x)$, and show that it finds the unique root of $x = g(x0$ in this case.

References.

1 Jonathan M. Borwein & Peter B. Borwein, "Pi and the AGM", Wiley, 1987.

2 B. W. Conolly, Solution of Problem 6672, *Amer. Math. Monthly* 100 (1993), 803.

3 John Todd, "Basic Numerical Mathematics, Vol. 1 Numerical Analysis", Academic Press, 1979.