Workshop 6, Math 311

1. Let (x_n) be a sequence of real numbers. Let $x \in \mathbf{R}$ and $m > 0$.

a. Show that the following two statements are equivalent:

(I) For every $\epsilon > 0$, there exists a $K(\epsilon)$ such that for every $n > K(\epsilon)$ we have $|x_n - x| < \epsilon$.

(II) For every $\epsilon > 0$, there exists a $K(\epsilon)$ such that for every $n > K(\epsilon)$ we have $|x_n - x| < m\epsilon$.

b. Use your result from a. to give a simpler proof of: the sum of two convergent sequences is convergent.

2. In this problem, we will calculate approximations to $\sqrt{3}$ using the method outlined in lecture.

a. Construct the following sequence: $s_1 = 1$, $s_{n+1} := (s_n + 3/s_n)/2$ for $n \geq 2$. Calculate the first ten terms of this sequence. Do you notice the sequence stabilizing?

b. How many terms do you need to calculate in order to approximate $\sqrt{3}$ with error less than $1/1000$?

c. Repeat a. and b. for different values of s_1 ($s_1 = .1$, 1.7, 10). What do you notice about the behavior of s_n in each case?

3. Let (x_n) be a bounded sequence. For each $n \in \mathbf{N}$ let

$$s_n := \sup\{x_k \ : \ k \geq n\}$$

$$t_n := \inf\{x_k \ : \ k \geq n\}$$

a. Calculate s_n and t_n for the following sequences:

(i) $x_n = (-1)^n/n$

(ii) $y_n = \begin{cases} 2 - 1/n, & n \text{ even;} \\ -2 + 1/n, & n \text{ odd.} \end{cases}$

b. Are s_n, t_n monotone? bounded? convergent?

c. Prove that for any choice of bounded sequence x_n, the corresponding sequences s_n and t_n are monotone and convergent.

d. Prove: if $\lim(s_n) = \lim(t_n)$ then (x_n) is convergent. Is the converse true? If so, prove it. If not, find a counterexample.

Note: $\lim(s_n)$ is called the **limit superior** of (x_n). $\lim(t_n)$ is called the **limit inferior** of (x_n).