Workshop 3, Math 311

1. a. Prove that the set of even natural numbers is countable.
b. Prove that the set of even integers is countable.
c. Prove that the set of irrational numbers is uncountable.

2. Let A_n, $n \in \mathbf{N}$ be an infinite sequence of sets. Recall that

$$\cap_{n\in\mathbf{N}} A_n := \{x \ : \ x \in A_n \ \ \forall n \in \mathbf{N}\}.$$

a. Let $I_n = [0, 1/n]$, for $n \in \mathbf{N}$. Prove that $\cap_{n\in\mathbf{N}} I_n = \{0\}$.
b. Let $J_n = (0, 1/n)$, for $n \in \mathbf{N}$. Prove that $\cap_{n\in\mathbf{N}} J_n = \emptyset$.
c. Let $K_n = [n, \infty)$, for $n \in \mathbf{N}$. Prove that $\cap_{n\in\mathbf{N}} K_n = \emptyset$.
d. Do the examples b. and c. contradict the Nested Interval Theorem? Why or why not?

3. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbf{R}$ be bounded. Let $F : X \to \mathbf{R}$ and $G : Y \to \mathbf{R}$ be defined by

$$F(x) := \sup\{h(x, y) : y \in Y\} \qquad G(y) := \sup\{h(x, y) : x \in X\}$$

a. Prove the **Principle of the Iterated Suprema:**

$$\sup\{h(x, y) : x \in X, \ \ y \in Y\} = \sup\{F(x) : x \in X\} = \sup\{G(y) : y \in Y\}.$$

b. Is it necessary for h to be bounded? If not, prove the relation for h unbounded. If so, give an example for which this equality does not hold.