Advanced Calculus I 311

Review Exercises

1. Let (x_n) be a sequence of real numbers. Determine whether each of the following equivalences is true or false. Where applicable, give a counterexample for the direction that is false.
   a. (x_n) convergent if and only if (x_n) monotone and bounded.
   b. (x_n) Cauchy if and only if (x_n) Cauchy.
   c. (x_n) convergent if and only if every subsequence of (x_n) is convergent.
   d. (x_n) is Cauchy if and only if (x_n) bounded.

2. Determine whether each of the following sequences is convergent or divergent. Prove your claim. You *must* use each of the following theorems or techniques at least once: definition of convergence ($\epsilon - K(\epsilon)$), Monotone Convergence Theorem, Divergence Criterion, Cauchy Convergence Criterion.
   a. $x_n = \frac{1}{n^2+a}$ where $a > 0$
   b. $y_n = \cos(n)$
   c. $z_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$
   d. $t_1 = 1$, $t_n = \sqrt{3 + t_{n-1}}$ for $n \geq 2$
   e. $s_n = \begin{cases} 2 - 1/n & \text{if } n \text{ even} \\ -2 + 1/n & \text{if } n \text{ odd} \end{cases}$

3. Let $I = [0, 1]$.
   a. Prove that the set of cluster points of $I \cap \mathbf{Q}$ is the set I.
   b. Prove that the point 5 is *not* a cluster point for I.

4. Determine whether each of the following functions has a limit or not at the indicated point. Prove your claim. You must give a proof using each of the following techniques at least once: $\epsilon - \delta$, sequential argument, Squeeze Theorem, Divergence Criterion
   a. $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ where $c > 0$
   b. $\lim_{x \to 1} \frac{2x-2}{3(x^2-1)} = \frac{1}{3}$
   c. $\lim_{x \to 0} \cos(1/x)$
   d. $\lim_{x \to 0} x \cos(1/x)$

5. Both of the following functions is not defined at $x = 1$. Can either of them be defined at $x = 1$ in such a way that they become continuous on all of $\mathbf{R}$? Prove your claims.

$$f(x) = \frac{x-1}{x^2 - x} \qquad g(x) = \frac{1}{x-1}$$

6. Prove that $x_n = \frac{1}{3^n}$ is a contractive sequence but $y_n = 1/n$ is not.

7. Let $f : \mathbf{R} \to \mathbf{R}$. Let $|f|(x) := |f(x)|$.
   a. Prove that if $\lim_{x \to c} f(x)$ exists then $\lim_{x \to c} |f|(x)$ exists .
   b. Is the converse true? If so, prove it. If not, give a counterexample.