Advanced Calculus I 311

Review Exercises

1. Prove the following assertions about a, $b \in \mathbf{R}$. You must give reasons for every step!
   a. $-(a+b) = -a + -b$
   b. $-(a/b) = (-a)/b = a/(-b)$ if $b \neq 0$
   c. $a \cdot a = 1$ if and only if $a = -1$ or $a = 1$
   d. If $0 < a < 1$ then $0 < a^n < a^m < 1$ for all n, $m \in \mathbf{N}$ such that $n > m$.

2. Prove that $\sqrt{5}$ is an irrational number.

3. Find all $x \in \mathbf{R}$ such that $|x-1| + |x+2| = 5$.

4. Let $I = [0,1]$.
   a. Prove that for all $\epsilon > 0$, the neighborhood $V_\epsilon(0)$ is *not* contained in I.
   b. Prove that for all $a \in (0,1)$, there exists an $\epsilon > 0$ such that $V_\epsilon(a)$ is contained in I.

5. Let A and B be bounded subsets of $\mathbf{R}$.
   a. Prove that $A \cup B$ is a bounded set.
   b. Prove that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

6. Let A be a nonempty bounded subset in $\mathbf{R}$. Let $b < 0$. Let $bA := \{ba \ : \ a \in A\}$. Show that

$$\inf(bA) = b(\sup A).$$

7. Let $S := \{1/n - 1/m \ : \ n, m \in \mathbf{N}\}$. Find $\inf S$ and $\sup S$.

8. Let $I_n = (-1/n, 1/n)$ for $n \in \mathbf{N}$. Show that 0 is the only real number that belongs to all I_n.

9. Prove that the set of odd numbers is countable.