Advanced Calculus I
Extra Credit Project — Polynomial Approximations

Please write all answers on separate sheets of paper. Your answers should be *numbered* and in the *same order* in which the problems appear. Your project should be *stapled* and your name should appear on *every* sheet.

Introduction: In this project you will learn about different techniques of approximating continuous functions by polynomials. You will use MAPLE to help you calculate various approximations and graph them.

The Approximation Problem: Given a continuous function $f : [0,1] \to \mathbf{R}$ and an $\epsilon > 0$, we wish to find a polynomial p such that

$$|f(x) - p(x)| < \epsilon \qquad \text{for all } 0 \le x \le 1.$$

In other words, we wish to approximate f by p in such a way that the error $f(x) - p(x)$ at each x is *at most* ϵ. The Weierstraß Approximation Theorem guarantees the *existence* of such polynomials, but doesn't indicate how one would construct them. We will explore various solutions to this problem for the function $f(x) = |x - 1/2|$ on the interval $[0,1]$.

1. Use MAPLE to plot $f(x)$. Use the command

>**plot(abs(x-1/2),x=0..1);**

Remember that the > is the MAPLE prompt. The command that you type is the boldface text *after* the prompt.

Part I - Connect the Dots: A "naive" appoach to the approximation problem is to just pick points $(x_1, f(x_1))$, $(x_2, f(x_2)), \ldots (x_n, f(x_n))$ and find the polynomial that connects those points. For example, if you want to approximate $f(x)$ by a parabola $p(x) = a_2x^2 + a_1x + a_0$ (that is, a polynomial of degree 2), you will need three distinct points $(x_1, f(x_1))$, $(x_2, f(x_2))$ and $(x_3, f(x_3))$ in order to determine $p(x)$.

2. Find the parabola $p(x)$ which passes through the points $(0, 1/2)$ and $(1/2, 0)$ and $(1, 1/2)$. In other words, you want to find a_2, a_1 and a_0 such that

$$\begin{aligned} p(0) &= a_2 \cdot 0^2 + a_1 \cdot 0 + a_0 = 1/2 \\ p(1/2) &= a_2 \cdot (1/2)^2 + a_1 \cdot 1/2 + a_0 = 0 \\ p(1) &= a_2 \cdot 1^2 + a_1 \cdot 1 + a_0 = 1/2 \end{aligned}$$

3. Use MAPLE to plot $p(x)$ along with $f(x)$. You can plot them all on one set of axes. If, for example, you obtained the result $p(x) = 2x^2 - 5x + 3$, then you would type

>**plot([abs(x-1/2), 2*x^2 - 5*x + 3], x=0..1);**

4. By looking at the graph, determine for which x is $p(x)$ a good approximation to $f(x)$. If $\epsilon = 1$, do we have $|f(x) - p(x)| < \epsilon$ for all $0 \le x \le 1$? What if $\epsilon = .1$ or $.01$?

It is natural to assume that one can get a better approximation simply by picking more points and finding the polynomial that goes through all of them. Such a polynomial is called an *interpolation polynomial*. There is a command in MAPLE that creates this polynomial for you. All you have to do is supply the points. For example, if you want the polyomial of smallest degree that goes through $(0, 1/2)$, $(1/4, 1/4)$, $(1/2, 0)$, $(3/4, 1/4)$ and $(1, 1/2)$, type:

>p5:=interp([1, 1/4, 1/2, 3/4, 1],[1/2, 1/4 , 0, 1/4, 1/2],x);

where $[1, 1/4, 1/2, 3/4, 1]$ represents all of the x values and $[1/2, 1/4, 0, 3/4, 1/2]$ represents the corresponding $f(x)$ values. The x at the end tells MAPLE that the interpolation polynomial should be a function of the variable x.

5. Using MAPLE, calculate $p5$ and find interpolation polynomials with 8, 10, 20 points. You can choose points that are evenly spaced (as for $p5$) or make up your own set of points. Assign each of these polynomials a name (such as we did for $p5$). In order to avoid making vectors with 10 or 20 entries by hand, you can use the following commands:

>X:=[seq((i-1)/19, i = 1..20)]; Y :=[seq(abs((i-1)/19 - 1/2), i=1..20)];

Then you can create your interpolation function (which we will call here $p20$) by typing:

>p20 := interp(X,Y,x);

6. Using Maple, plot $p, p5, p8, p10$ along with $f(x)$ on the same axes:

>plot([abs(x-1/2),p,p5,p8,p10],x=0..1);

7. Now include $p20$:

>plot([abs(x-1/2),p,p5,p8,p10,p20],x=0..1);

8. What happens as you use more and more points? What is the error for each of the interpolation polynomials?

As you can see from this experiment, the problem of choosing a good approximation to $f(x)$ is more subtle than simply finding a polynomial that shares some of the same values as f.

Part II - Bernšteĭn Polynomials:†

Definition: Let $f : [0, 1] \to \mathbf{R}$ be continuous. The n^{th} *Bernšteĭn polynomial for f* is the polynomial $B_n(x)$ defined by

$$B_n(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

† S. N. Bernšteĭn (born 1880), Russian mathematician

Notice that in order to calculate $B_n(x)$ one must evaluate f at the $n+1$ evenly spaced points $0, 1/n, \ldots, k/n, \ldots, (n-1)/n, 1$ in the interval $[0,1]$.

9. Calculate the first six Bernšteĭn polynomials for $f(x) = |x - 1/2|$ on the interval $[0,1]$.

10. Using MAPLE, plot B_1, B_2, B_4 and B_6, along with $f(x)$ on the same set of axes. You might first want to assign the Bernšteĭn polynomials names such as $b1$, $b2$, $b4$, $b6$. For example, if $B_1(x) = 1/2$ then type

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>b1 := 1/2;
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Then, in order to plot the graphs, type

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>plot([abs(x-1/2),b1,b2,b4,b6],x=0..1);
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11. What is the error of approximation for each Bernšteĭn polynomial?

12. We now use MAPLE to calculate a few more Bernšteĭn polynomials. MAPLE has a built-in function called **bernstein** which does just that. First type

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>readlib(bernstein);
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then type

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>b10 := bernstein(10, x -> abs(x-1/2),x);
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This will assign **b10** the value of $B_{10}(x)$. Repeat this for various n (n = 15, 20, 25, 30). You can print out the results and paste them into your project (instead of handwriting them in).

13. Plot all Bernšteĭn polynomials that you calculated along with $f(x)$ on the same set of axes. What happens as n becomes larger?

14. By looking at the graph, determine at which point x_0 is the error $|f(x) - B_n(x)|$ the greatest.

15. Calculate $B_n(x_0)$ for all n. The point x_0 is the same one that you just found. Hint: You will need to use the binomial identity:

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$$

16. We now assume that the results from questions 14 and 15 are true ‡. In other words, that

$$|f(x) - B_n(x)| \leq |f(x_0) - B_n(x_0)| \qquad \forall n \ \ 0 \leq x \leq 1.$$

where $B_n(x_0)$ is the same as in question 15. Prove that the sequence $(B_n(x))$ converges uniformly to $f(x)$ on the interval $[0,1]$.

‡ Note that *proving* these results is <u>no</u> trivial matter.

17. How large does n have to be in order for $B_n(x)$ to approximate $f(x)$ with an error of at most .001?

As you can see from these exercises, the Bernšteĭn polynomials are a much better approximation to $f(x) = |x - 1/2|$ than the interpolation polynomials. In general, Bernšteĭn polynomials can be used to closely approximate *any* continuous function on a closed interval:

Theorem (Bernšteĭn): *If f is continuous on $[0, 1]$, then its Bernšteĭn polynomials B_n converge uniformly to it on $[0, 1]$ as $n \to \infty$.*

A Little History: So how did Bernšteĭn come up with such polynomials? He probably knew a bit of probability theory. Suppose that one has a coin with the property that the probability of its showing heads after a single toss is x $(0 \leq x \leq 1)$. Then the probability of its showing tails after one toss is $1 - x$. Moreover, the probability of obtaining exactly k heads in n tosses is

$$\binom{n}{k} x^k (1-x)^{n-k}.$$

Since we must obtain *some* number of heads from 0 to n in n tosses of the coin, we have: probability of getting 0 heads + probability of getting 1 head + ... + probability of getting n heads =

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$$

Now, think of $f(k/n)$ representing the amount of money that you win (or lose) if you toss exactly k heads. Then

$$\sum_{k=0}^{n} \binom{n}{k} f(k/n) x^k (1-x)^{n-k}$$

represents the *expected value* that you will win with n tosses. It is a consequence of a theorem in probability theory called *The Law of Large Numbers* that as n becomes larger, the expected value approaches $f(x)$.