Intro to Mathematical Reasoning (Math 300)

Supplement 1. **Universal Statements**[1]

This supplement builds on the material discussed in section 1.1-1.3 of the textbook. It is assumed that you have read these sections before reading the supplement.

Universal statements

Section 1.3 of the textbook introduces the *universal quantifier* and the *existential quantifier*. Here we discuss these in a little more detail.

Consider the following two statements:

Statement A. $(7-3)(7+3) = 7^2 - 3^2$

Statement B. For any two numbers a and b, $a^2 - b^2 = (a+b)(a-b)$.

The first is a statement about the specific numbers 3 and 7. The second is a much more interesting and powerful statement than the first, because it asserts a *general principle* that works for any two numbers you choose.

We call such a statement a *universal statement*. This universal statement implies the first statement, and an endless number of others, such as:

Statement D. $(7101036 - 3354251)(7101036 + 3354251) = 7101036^2 - 3354251^2$

Statements A and D are said to be *instances* of the universal statement B. They are obtained by substituting specific values for a and b. Since, as we all know, statement B is a true statement, it tells us that statement D must be true also. Notice that statement D is not something that would be easy to verify directly (even with a calculator).

Universal statements are the most important type of statement, not only in mathematics but in any field of study. Universal statements allow us to summarize *vast amounts of knowledge* in a *single sentence.*

Let's look at another example. Recall that if a and b are integers we say that b *is a divisor of* a or a *is divisible by* b or a *is a multiple of* b if there is an integer that when multiplied by b gives a. We say p is a *prime number* if it is an integer greater than 1 and has no positive divisors other than 1 and itself.

Statement E. Every prime number is odd.

What are instances of this statement? There are no variables to substitute for, but we can rewrite the statement as:

Statement E'**.** For every prime number n, n is odd.

Here are two statements.

Statement F**.** 17 is odd.

Statement G**.** 2 is odd.

Statement F is an instance of Statement E. Since 17 is a prime number, we can substitute 17 for n. So if Statement E is true then Statement F must be true.

Since 2 is prime, we can substitute 2 for n. Therefore, if E is true then G must be true. But, of course, G is false since 2 is not odd.

From this we conclude that E must be a false statement.

This example illustrates the following fundamental two part principle:

- A universal statement is *true* provided that *all of its instances* are true.
- If a universal statement has *even one false instance*, then it is *false*.

A true instance of a universal statement is said to be an *example* of that statement. A false instance of a universal statement is said to be a *counterexample* of that statement.

Here is one more illustration of a universal statement:

Statement I**.** For every positive integer n, and any prime number a, a is a divisor of $n^a - n$.

Here are some instances:

Statement J**.** 5 is a divisor of $4^5 - 4$.

Statement K**.** 3 is a divisor of $8^3 - 8$.

You can check that both of these are true statements. However, checking that a few instances are true, does not show that statement I is true. Statement I has infinitely many instances, and for I to be true, all of its instances must be true. So all we can say is that statements J and K *illustrate* statement J but they do not *prove* statement K.

It turns out that statement I really is true. If there is time in this course, we may prove it, but for now we will leave you to ponder this amazing fact. Before leaving this example consider another statement:

Statement L**.** 4 is a divisor of $5^4 - 5$.

This is a true statement since 4 is a divisor of $5^4 - 5 = 3120$. Is it an instance of Statement I?

The answer is no. Statement I says that if you substitute any integer for n and any prime number for a then a divides $n^a - n$. But 4 is not prime! Therefore, while statement L is true, it does not follow from statement I.

Making a false universal statement true by restricting the hypothesis

If a universal statement is false, this means that some instances are false. But it may be that the conclusion is still true for many instances. It is often possible to modify a false universal statement to a true one by modifying the hypothesis so as to eliminate all of the false instances. For example, we can modify statement E to get:

> **Statement E''.** Every prime number other than 2 is odd.

Here we have modified our statement to specifically exclude 2, and now there are no counterexamples. So statement E'' is true.

Here's another example. Consider the statement:

> **Statement Q.** For any three real numbers a, b, c that satisfy $ab = ac$ we must have $b = c$.

At first glance this seems to be a true statement, one that you learned many years ago. However, if you think carefully you'll realize that it is not true. Consider the instance $a = 0$, $b = 4$ and $c = 1$. Then $ab = ac = 0$ but $b \neq c$.

Once we realize that this is false, we can also see that if we modify the hypothesis a little bit then we can make it true.

> **Statement R.** For any three real numbers a, b, c that satisfy $ab = ac$ and $a \neq 0$ we must have $b = c$.

Notice that in both examples, we modify the statement so that the bad cases don't satisfy the hypothesis. This is called *restricting the hypothesis.*

When we restrict the hypothesis to correct an incorrect universal statement, we want to restrict it as little as possible. Another way to restrict statement R to make it correct is to write:

> **Statement S.** For any three positive integers, a, b, c that satisfy $ab = ac$ we must have $b = c$

Statement S is true, but statement R is preferable because it is also true, but it puts fewer restrictions on the hypothesis. Statement R covers all of the instances of statement S and many more, such as the instance $a = -4.5$, $b = 17.3$ and $c = \pi$.

The structure of universal statements

Universal statements take the following general form:

> Any choice of a mathematical object (or objects) that together satisfy condition (or conditions) H, must also satisfy condition C.

The condition or conditions H is the *hypothesis* of the universal statement while condition C is the conclusion. In statement I, the mathematical objects are represented by the letters a and n and the hypothesis and conclusion are:

Hypothesis: n is a positive integer and a is prime.

Conclusion: $n^a - n$ is a multiple of a.

The hypothesis and conclusion are *mathematical predicates*. Recall that a mathematical predicate is a sentence that becomes a mathematical statement if you substitute values for the variables, where the statement obtained may be true or false depending on the values substituted.

It is customary to use the notation $H(a, n)$ to mean that H is a predicate expressed in terms of two variables a and n. So the form of statement I is:

All choices of a and n that make $H(a, n)$ true must also make $C(a, n)$ true.

We sometimes refer to the variables a and n as the *input variables* for the statement. (This is not standard mathematical terminology.)

We can now say more carefully what we mean by an instance, an example and a counterexample to a universal statement.

1. An *instance* of a universal statement is an assignment of values to the input variables that make the hypothesis true.

2. An *example* of a universal statement is an assignment of values to the input variables that make the hypothesis true and the conclusion true.

3. A *counterexample* for a universal statement is an assignment of values to the input variables that make the hypothesis true, together with an conclusion is false.

Notice:

- Every instance is either a counterexample or an example but not both.

- An assignment of the input variables that makes the hypothesis *false* is neither an instance, example nor a counterexample.

Let's consider another example of a statement.

Statement M**.** For every prime n such that $n - 1$ is divisible by 4, it is possible to find two positive integers such that n is equal to the sum of their squares.

Here there are three hypotheses "n is a positive integer", "n is prime", and "$n - 1$ is divisible by 4". The conclusion is "It is possible to find two positive integers such that n is equal to the sum of their squares."

29 provides an *instance* of the universal statement because it satisfies the hypothesis. Is it an example or a counterexample. To figure this out requires some work. Can we find two positive integers whose squares sum to 29? A bit of searching shows us the answer is yes: $29 = 5^2 + 2^2$.

If asked, "is 29 an example or counterexample to statement M?" it is not really enough to say simply "29 is an example". Rather you should include an explanation:

29 is an example of statement M because 29 is prime and 29-1 is divisible by 4 and so 29 satisfies the hypothesis, and $29 = 5^2 + 2^2$

Other ways to express universal statements

Statement A'. For any choice of two numbers, the square of the first number minus the square of the second is equal to the number obtained by subtracting the second number from the first and multiplying the result by the sum of the two numbers.

This is a universal principle. If you think about it, Statement A' is just a restatement of Statement A. Statement A is shorter and easier to read because it makes use of letters to represent the mathematical objects involved.

Even when using letters, there are different ways to express a universal principle. Consider:

1. Every prime number greater than 2 is odd.
2. Each prime number greater than 2 is odd.
3. For all prime integers n that are greater than 2, n is odd.
4. For all prime integers n satisfying $n > 2$, n is odd.
5. For all integers n such that n is prime and $n > 2$, n is odd.
6. For all prime integers n, if $n > 2$ then n is odd.
7. For all integers n, if n is prime and $n > 2$ then n is odd.

Look at all of these carefully. All of them express the same universal principle in different language. The words "every" and "for all" clue you in that this is a universal statement. The words "satisfying" and "such that" after the "For all n" is a clue that what follows is part of the hypothesis. Also, in the last two examples, the part after "if" is part of the hypothesis and the part after "then" is part of the conclusion.

Important Note: In the last sentence above, mathematicians will often omit the initial "For all integers n" and write simply:

If n is prime and $n > 2$ then n is odd.

Technically speaking this does not mean the same as the above statements because it is a mathematical predicate not a statement. Thus it is technically incorrect to write this sentence when you mean one of the others. Yet mathematicians do it all the time and when experienced mathematicians see the above sentence they know that the writer intended that there to be an invisible "For all n" in the front. So for experienced mathematicians, this is an acceptable shortcut.

The textbook often uses this shortcut. For example, the second example on page 32 says "Prove that if $x < -4$ and $y > 2$ then the distance from (x, y) to $(1, -2)$ is at least 6." Here the technically correct way to express this is "Prove that for all real numbers x, if $x < -4$ and $y > 2$ then the distance from (x, y) to $(1, -2)$ is at least 6."

However, for most students just learning about proofs this can be confusing. Dropping the "For all n" can lead a student to serious mistakes in writing proofs. Experienced mathematicians know enough to avoid such mistakes, but many beginners don't.

Therefore, for now, it is recommended that students in this course *should always include* the initial "For all" portion of a universal statement, even though the textbook often omits the "For all" part.

Finally, let's mention that there is a symbolic abbreviation for "For all" which is "$\forall$". Using this abbreviation one can restate the sentences above as.

> $\forall n$, if n is prime and $n > 2$, then n is odd.

The "$\forall$" symbol is called the *universal quantifier.*

Vacuously true statements

Consider the statement:

> **Statement T.** For every pair of positive integers n and m, if $n \geq m + 3$ and $m^2 \geq n^3$ then $m + n$ is prime.

The input variables for the universal statement are n and m and the hypothesis is that n and m are positive integers, $n \geq m+3$ and $m^2 \geq n^3$. A bit of thought shows that there are no assignments to m and n that satisfy this hypothesis. For, imagine that m and n are positive integers that do satisfy the hypothesis. Then $n \geq m + 3$ implies $n > m$ which implies $n^2 > m^2$. But since $m^2 \geq n^3$ we can combine this with $n^2 > m^2$ to conclude $n^2 > n^3$. Dividing both sides by n^2 gives $1 > n$ which is impossible since n is a positive integer.

This means that every choice of n and m makes the hypothesis false. What are we to conclude about statement T?

Keep in mind that a universal statement is true provided that every assignment of the input variables that satisfies the hypothesis also satisfies the conclusion. Since there are no assignments that satisfy the hypothesis, then T is TRUE.

Another way to look at it is that for T to be false, there must be a counterexample, but if nothing satisfies the hypothesis then there can be no counterexamples.

A statement that is true because no assignments satisfy the hypothesis is said to be *vacuously true.*

The negation of a universal statement

Page 5 and 6 of the book introduces the notion of the negation of a proposition, and there is further discussion of this in on pages 21-23.

The negation of statement E above is:

Statement N**.** It is not the case that every prime number is odd.

As is discussed in the book, we can often construct an equivalent sentence that is not in the form of a negative sentence. A sentence that is equivalent to the negation of a sentence is called a denial of a sentence.

A sentence beginning with the phrase "It is not the case that . . . " or something similar is called a *negative sentence*. How do we construct a statement that is equivalent to N so that it is not in the form of a negative sentence? When we say that the sentence "All prime numbers are odd" is not true we mean that there is at least one counterexample to the sentence, that is, at least one object that satisfies the hypothesis (is a prime number) but does not satisfy the conclusion (is not odd). So a sentence with the same meaning as the negation is:

Statement O**.** There is at least one prime number that is not odd,

These are examples of *existential statements*, which are statements that assert only that there is at least one object satisfying certain conditions. So the general form is:

> It is possible to find a mathematical object (or objects) that satisfies condition (or conditions) P.

As with universal statements, there are many equivalent ways to restate an existential statement. Here are some possible ways to restate the statement above:

1. There is at least one number that is prime and not odd.
2. For some prime number n, n is not odd.
3. There exists a number that is prime and not odd.
4. There exists a number n satisfying the conditions that n is prime and n is not odd.
5. There exists a number n such that n is prime and n is not odd.

All of the above statements are perfectly good restatements of statement N. The phrases "There is" and "There exist" and "For some" are the clue to this being an existential statement. The phrase "satisfying the conditions that" or "such that" clue you in that what follows is the condition or conditions that are required.

One point: It was stated earlier that the phrase "every" or "for all" indicates that a statement is a universal statement. However, if "every" or "for all" is preceded by "It is not the case that" as in statement N above, the statement is no longer a universal statement. As we've seen, it is equivalent to an existential statement.

Proving and disproving universal statements

To prove a mathematical statement means, informally, to provide a completely convincing argument that the statement is true. To disprove a mathematical statement means, informally, to provide a completely convincing argument that its negation is true.

It is not at all clear how to go about proving a universal statement that you think is true. If I give you a statement such as Statement I, then you can check it by trying lots of instances, and they will all work. But this does not prove statement I is true. Just because every instance you try supports the statement, does not show that every instance will work.

Nevertheless, it is possible to prove many universal statements, and that is the main thing you will learn in this course. But not just yet ...

On the other hand, if a universal statement is *false*, then disproving it is often very easy. To disprove a universal statement you only need to present a single counterexample, and demonstrate that it is a counterexample.

Here's an example:

Disprove the statement: For every choice of integers a,b,c,d, if a, b, c, d are different then $a^2 + b^2 \neq c^2 + d^2$.

Disproof: Since this is a universal statement it is enough to find a counterexample and demonstrate that it is a counterexample. Consider the instance $a = 8$, $b = 1$, $c = 7$, $d = 4$, which satisfies the hypothesis. Since $8^2 + 1^2 = 65 = 7^2 + 4^2$, the conclusion is not satisfied and so this is a counterexample. Thus the statement is disproved.