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LAB 2: Linear Equations and Matrix Algebra

In this lab you will use MATLAB to study the following topics:

- Solving a system of linear equations by using the reduced row echelon form of the augmented matrix of the system.
- Forming linear combinations of a set of vectors and the fundamental concepts of *linear dependence* and *linear independence*.
- Matrix multiplication and its properties.
- Applications of $(0, 1)$ Matrices

Preliminaries

Reading from Textbook: Before beginning the Lab, read through Sections 1.3, 1.4, 1.6, 1.7, and 2.1 of the text and do the suggested homework problems for these sections.

MATLAB **Help:** In Lab 1 you learned the basic MATLAB commands. Remember that every MATLAB command is documented in a `help` file, which you can access during a MATLAB session. For example typing `help format` gives information about the command `format`. Look in Appendix D of the text and go to the mathematics department web site for this course for additional MATLAB documents if you want further information. For this lab, create a `diary` file and edit it as you did for Lab 1.

Script Files: For more complicated MATLAB calculations you should use *scripts*. A script contains one or several MATLAB commands and is stored as a text file with a descriptive name such as `mymatrix.m`, for example (the extension `*.m` is required). When you type the name of a script (without the extension `*.m`) at the MATLAB command prompt the commands within the script are executed, affecting the variables in the global workspace. Such scripts are called *m-files*. The advantage of having scripts is that you can execute the commands in the script at any time by typing the *name* of the script instead of the *contents* of the script.

Writing Scripts: When you need to write a script file for this lab and subsequent labs, use the following procedure: Start MATLAB and click on *File*, then *New*. Move the pointer to the right and click on *Script*. This will open the MATLAB Editor/Debugger Window, and you can type the script commands in this window. You can take any m-file, edit it (just as you would edit any text file), and then save it under a different name to obtain a new m-file.

Running Scripts: After you have created an m-file and saved it to your directory, you must set the *Path* so that MATLAB can find this file. Click on *File*, then *Set Path* and follow the directions to add your directory to the list of path names.

Script Files for Lab 2: Use the text editor in MATLAB to create the following MATLAB *function* m-files:

(a) **`rvect.m`:** Create a *function* m-file with the commands

```
function v = rvect(m)
v = fix(10*rand(m,1));
```

(note the semicolon on the end of the second line). Save this file under the name `rvect.m` (be sure that you have set the *Path* as described above so that MATLAB can find this m-file). Test the file by clicking on the MATLAB window and typing `v = rvect(3)` at the MATLAB prompt. You should get a column vector $\mathbf{v} \in \mathbf{R}^3$ with entries that are (random) integers between 0 and 9. Now type `u = rvect(3)`. You will get another random column vector $\mathbf{u} \in \mathbf{R}^3$. Type `v` at the prompt. You should get the *same* vector $\mathbf{v}$ as before.

Note that the name **v** in the `rvect` function file is a *local variable*; you can assign any name to the output. If you have already defined a vector **v** in your work space, it is not changed when you generate **u** by `rvect`.

(b) **`rmat.m`:** Create another function m-file with the commands

```
function A = rmat(m,n)
A = fix(10*rand(m,n));
```

(note the semicolon on the end of the second line). Save this file under the name `rmat.m`. Then test the file by clicking on the MATLAB window and typing `A = rmat(3, 5)` at the MATLAB prompt. You should get a 3×5 matrix A with entries that are (random) integers between 0 and 9.

Lab Write-up: After writing and testing the script files, open your diary file (see Lab 1 for details). Begin the diary file with the following comment lines, filling in your information where appropriate.

```
% [Name]
% [Last 4 digits of RUID]
% Section [Section number]
% Math 250 MATLAB Lab Assignment #1
```

Type `format compact` so that your diary file will not have unnecessary spaces. Type `format short` so that the numbers will appear to four decimal places. Put labels to mark the beginning of your work on each part of each question. For example,

```
% Question 1 (a) ...
        ⋮
% Question 1 (b) ...
```

and so on.

Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment.

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

10

Question 1. Solving $A\mathbf{x} = \mathbf{b}$

In this question you will find the general solution $\mathbf{x} \in \mathbf{R}^3$ to a linear system $A\mathbf{x} = \mathbf{b}$ of 3 equations in 5 variables x_1, x_2, x_3, x_4, x_5. Here A is the 3×5 coefficient matrix and $\mathbf{b} \in \mathbf{R}^3$ is the given right-hand side of the system.

Random Seed: Initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your student ID number. This will ensure that you generate your own particular random vectors and matrices. BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP.

(a) Use your MATLAB function files to generate a random 3×5 integer matrix A and to check that columns 1, 2, 3 of A are the pivot columns.

```
A = rmat(3, 5), rank(A(:,1:3))
```

Note the use of the colon operator to select columns 1, 2, 3 of A. If the rank is less than 3, generate a new A (this is unlikely, but it can happen). Include all the matrices that you generate this way in your lab report. When you have an A for which the rank of the first three columns is 3, then generate a random vector $\mathbf{b} \in \mathbf{R}^3$ and the reduced row echelon form R of the augmented matrix $[A\ \ \mathbf{b}]$:

```
b = rvect(3), R = rref([A  b])
```

To get the reduced row echelon form $S = \text{rref}(A)$ just remove the last column from R:

```
S = R(:,1:5)
```

(Note the use of the colon operator to select columns 1 to 5 of R). Check by MATLAB that `S = rref(A)`; then type answers to the following using comments:

(1) *(i)* Which columns of S are the *pivot columns*?

(1) *(ii)* What is the *rank* of R and the *rank* of A?

(1) *(iii)* What is the *nullity* of A and which variables x_i are the free variables?

(1) *(iv)* Why does the equation $A\mathbf{x} = \mathbf{b}$ have a solution?

(b) Use MATLAB to obtain `c = R(:,6)` (the last column of `R`), and set `x = [c; 0; 0].` Note carefully that semicolons appear in this formula before the zeros, so that $\mathbf{x} \in \mathbf{R}^5$ and the last two components of $\mathbf{x}$ are zero.

(1) *(i)* Calculate by MATLAB that `A*x - b = 0` and `S*x - c = 0`.
IMPORTANT: Your answer vectors here are probably not exactly equal to the zero vector. Because of the finite precision of computer arithmetic and roundoff error, vectors or matrices that are zero (theoretically) may appear in MATLAB in exponential form such as `1.0e-014 * M` (where `M` is a vector or matrix with entries between -1 and 1). An example of MATLAB output in this form is

```
ans =
   1.0e-14 *
    0.0222
    0.0888
    0.1776
```

which represents the vector $10^{-14} \cdot [0.0222, 0.0888, 0.1776]^T$. (NOTE: The first row in this answer, before the `*`, is not a component of a vector. It is a scalar multiplying the column vector appearing in the following three rows.) This means that each component of the answer is less than 10^{-14} in absolute value, so the vector or matrix can be treated as zero (numerically) in comparison to vectors or matrices with entries that are on the order of 1 in size. Whenever you are asked to verify by MATLAB that two matrices or vectors are equal, calculate their difference and use this meaning of "zero". (Here, answers such as `1.0 e-013 * M` or `1.0 e-012 * M` would also be considered as "zero".)

It is crucial that you remember this information about "zero" in the later Labs.

(1) *(ii)* Use properties of row reduction to explain why the equations in *(i)* are true. (See pages 33 and 127 of the text.)

(c) Use MATLAB to calculate `u = [-S(:,4); 1; 0],    v = [-S(:,5);0;1].`

(1) *(i)* Give a handwritten explanation, using *symbols* and linear algebra, rather than numbers, to show why $\mathbf{u}$ and $\mathbf{v}$ are the vectors that appear in the *vector form* of the general solution to $A\mathbf{x} = \mathbf{0}$.

(See page 80 of the text.) Confirm by calculating `S*u, A*u, S*v, A*v`. You should get vectors that are (approximately) zero.

Now generate a random linear combination of $\mathbf{u}$ and $\mathbf{v}$ by the commands

```
s = rand(1),   t = rand(1), y = s*u + t*v
```

(Note that each occurrence of `rand(1)` generates a different random coefficient).

(1) *(ii)* What properties of matrix and vector algebra ensure that $A\mathbf{y} = 0$? (Look at Theorem 1.3 on p. 24 of the text.)

Confirm by a MATLAB calculation that $A\mathbf{y}$ is approximately zero.

(d) Use MATLAB to calculate `z = x + y` .

(2) What properties of matrix and vector algebra imply that $A\mathbf{z} = \mathbf{b}$? (Look at Theorem 1.3 on p. 24 of the text.)

Confirm by a MATLAB calculation that $A\mathbf{z} - \mathbf{b}$ is approximately zero.

6 Question 2. Spanning Sets and Linear Independence

Generate four random vectors in $\mathbf{R}^3$ by the command

```
u1 = rvect(3), u2 = rvect(3), u3 = rvect(3), u4 = rvect(3)
```

Use these vectors in the following.

(a) Consider the set of vectors $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. To determine whether $\mathcal{S}$ is linearly independent, form the matrix A with the vectors from $\mathcal{S}$ as columns and calculate its reduced row echelon form:

```
A = [u1 u2 u3],   rref(A)
```

Use these calculations to answer the following questions:

(1) *(i)* How many free variables does the equation $A\mathbf{x} = \mathbf{0}$ have?

(1) *(ii)* Is the set $\mathcal{S}$ linearly independent or linearly dependent? Why?

(b) Consider the set of vectors $\mathcal{T} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$. To determine whether $\mathcal{T}$ is linearly independent, form the matrix B with the vectors from $\mathcal{T}$ as columns and calculate its reduced row echelon form:

```
B = [u1 u2 u3 u4],   rref(B)
```

Use these calculations to answer the following questions:

(1) *(i)* How many free variables does the equation $B\mathbf{x} = \mathbf{0}$ have?

(1) *(ii)* Is the set $\mathcal{T}$ linearly independent or linearly dependent?

(c) Let $\mathbf{v}$ be a random linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$:

```
v = rand(1)*u1 + rand(1)*u2
```

Thus $\mathbf{v}$ is of the form $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ for some scalars c_1, c_2. Consider the set of vectors $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$.

(1) Is the set $\mathcal{U}$ linearly independent or linearly dependent?

Answer first *without calculation* using the *definition* of linear independent sets.

(1) Check your answer by MATLAB using the method of part **(a)**.

4 Question 3. Matrix Multiplication

For this question generate random matrices and a random vector:

```
A = rmat(2,3), B = rmat(3, 4), C = rmat(4,3), v = rvect(4)
```

To obtain the product AB of the matrices A and B using MATLAB, you must type `A*B`. This is only defined when the number of columns of A is the same as the number of rows of B (you will get an error message when the matrix sizes are not compatible).

(1) **(a) Associativity:** The product AB is defined uniquely by the property that A applied to a vector $\mathbf{u} = B\mathbf{v}$ is the same as the matrix AB applied to the vector $\mathbf{v}$, for every vector $\mathbf{v}$ of the correct size. Verify this by calculating

```
u = B*v, A*u, D = A*B, D*v
```

This property implies the *associativity* of matrix multiplication: $A(BC) = (AB)C$. Verify this for the matrices A, B, C that you have generated.

(b) Matrices Are Not Numbers: Matrix algebra has many similarities to (ordinary) algebra of numbers, but there are important differences (which are the mathematical source of the differences between classical mechanics and quantum mechanics in physics). Here are some examples. Generate matrices

```
A = [0  1; 0  0],  B = [0  0; 1  0],  C = [0  1; 1  0]
```

Use these matrices and MATLAB calculations to answer the following:

(1) *(i)* Is $AB = BA$? Is $(A+B)^2 = A^2 + 2AB + B^2$? (Note that both of these equations would be true if A and B were numbers instead of matrices.)

(1) *(ii)* Calculate A^2. If A were a number instead of a matrix, what would the value of A^2 tell you about the value of A? Is this conclusion valid for matrices?

(1) *(iii)* Calculate AC and compare it with AB. If A, B, C were numbers with $A \neq 0$, what would you conclude about B and C from this calculation? Is this conclusion valid for matrices?

5 Question 4. $(0, 1)$ Matrices

Read the material on $(0,1)$ Matrices in **Section 2.2**, pp. 112–115 in the text and work through Practice Problem #3 on page 115. The following questions refer to Exercise #26 on page 121 of the text. You should work through parts (a)–(e) of Exercise #26 on your own (no writeup necessary). Then use hand calculations and Matlab to write up answers to the following questions.

Enter the 6×6 matrix A given before part (f) into your MATLAB workspace (be careful to copy A correctly).

(2) *(i)* On a separate page draw by hand six vertices labeled 1–6 (one for each person). Make this into a *directed graph* by putting an arrow pointing from vertex i to vertex j for all ordered pairs (i, j) for which the entry $a_{ij} = 1$ in A. Scan this graph and save as a .jpeg or .pdf file. This file should be uploaded along with your assignment when you do the Sakai submission.

Give detailed solutions in your lab report for the following.

(1) *(ii)* Answer part (f) of Exercise #26. Obtain the answer in two ways: by the graph and by the matrix A. Give details.

(1) *(iii)* Answer part (g) of Exercise #26. Use the matrices A, A^2, A^3, and A^4 to obtain your answer and consider the four cases (1, 2, 3, 4 stages) separately. Give details.

(1) *(iv)* Answer part (h) of Exercise #26. Use the matrix $B = A + A^2 + A^3 + A^4$ to obtain your answer. Give details.

Final editing of lab write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. Be sure to consult the instructions at the end of the MATLAB Demo assignment. Here is a summary:
Correct all typing errors and remove any unnecessary blank lines in your diary file. Your write-up must contain only the input commands that you typed which were required by the assignment (including `format compact` at the beginning), the output results generated by MATLAB, immediately following the corresponding input commands, your answers to the questions in the indicated places, and the indicated comments such as question numbers.
In particular, remove the commands `load, save, clear, format, help, diary`, **with the exception of** `format compact`, and remove any output from the commands `load, save, clear, format, help, diary`, as well.
Save the file as a plain text file.

Lab write-up submission guidelines: Preview the document before uploading and remove unnecessary page breaks and blank space. Make sure any images that need to be uploaded are in .jpeg or .pdf formats. Sakai will not allow you to upload files other than .pdf, .jpeg, or ,txt. Give yourself sufficient time to go through the submission procedure. Make allowances for computer and internet issues, as well as clock differences. *Late submissions will not be accepted.* Please be aware that both upload and submit steps need to be completed. If you do not complete both steps, your files will not be visible to the graders and you will receive a zero for the assignment.

Important: The submission of an unedited diary file without comments will be penalized by the removal of a significant number of points from the score.