(25) 1. Calculate the following limits. Give a brief justification of your answers without reference to calculator computations or graphing.

(a)
$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

First solution:

$$\lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x \to 3} x + 3 = 6$$

Second solution using L'Hôpital's Rule:

$$\lim_{x \to 3} \frac{2x}{1} = 6$$

(b) $\lim_{x \to 0} \frac{\sin 3x}{\tan 2x}$

First solution:

$$\lim_{x \to 0} \frac{\sin 3x}{\frac{\sin 2x}{\cos 2x}} = \lim_{x \to 0} \cos 2x \left(\frac{3}{2}\right) \frac{\frac{\sin 3x}{3x}}{\frac{\sin 2x}{2x}} = 1 \left(\frac{3}{2}\right) \left(\frac{1}{1}\right) = \frac{3}{2}$$

Second solution:

$$\lim_{x \to 0} \frac{\sin 3x}{\frac{\sin 2x}{\cos 3x}} = \lim_{x \to 0} \cos 2x \frac{\sin 3x}{\sin 2x} = \frac{3}{2}$$

Third solution using L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{3\cos 3x}{2\sec^2 2x} = \frac{3}{2}$$

(c)
$$\lim_{x \to \infty} \frac{5x^4 + 7x^3 + 2x^2 + 10}{7x^4 + 5x^3 + 2x + 1}$$

$$\lim_{x \to \infty} \frac{5 + \frac{7}{x} + \frac{2}{x^2} + \frac{10}{x^4}}{7 + \frac{5}{x} + \frac{2}{x^3} + \frac{1}{x^4}} = \frac{5}{7}$$

(d)
$$\lim_{x \to \infty} \frac{e^{x/2}}{x^2}$$

Applying L'Hôpital's Rule once, we get

$$\lim_{x \to \infty} \frac{\frac{1}{2}e^{x/2}}{2x},$$

which is still indeterminate. Applying the rule again, we get

$$\lim_{x \to \infty} \frac{\frac{1}{4}e^{x/2}}{2} = \infty.$$

Thus the original limit is ∞ .

(e)
$$\lim_{x \to 3} \frac{x^2}{\ln x}$$

$$\frac{3^2}{\ln 3}$$

(10) 2. Compute the derivative of $\sqrt{x+3}$ directly from the definition.

$$(\sqrt{x+3})' = \lim_{h \to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h+3} - \sqrt{x+3})(\sqrt{x+h+3} + \sqrt{x+3})}{h(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \lim_{h \to 0} \frac{x+h+3 - (x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}$$

- (25) 3. Compute the derivatives with respect to x of the following functions. Algebraic simplification of the answers need not be performed.
 - (a) $\ln(x)\cos(2x)$

$$\frac{\cos(2x)}{x} - 2\ln(x)\sin(2x)$$

(b)
$$\frac{e^x}{2x^3 + x}$$

$$\frac{(2x^3+x)e^x - e^x(6x^2+1)}{(2x^3+x)^2} = \frac{e^x(2x^3-6x^2+x-1)}{(2x^3+x)^2}$$

(c)
$$\int_0^x \sec t \, dt$$

By the Fundamental Theorem of Calculus, the derivative is

$$\sec x$$

$$(d) \int_0^{x^3} e^{t^2} dt$$

By the Fundamental Theorem of Calculus plus the chain rule, the derivative is

$$e^{(x^3)^2} 3x^2 = e^{x^6} 3x^2$$

(e)
$$\sqrt{x^4 + 3}$$

$$\frac{1}{2}(x^4+3)^{-1/2}4x^3 = \frac{2x^3}{\sqrt{x^4+3}}$$

(10) 4. Suppose that f is a function with first and second derivatives. Suppose in addition that the following values are known: f'(0) = 2, f'(1) = 3, f''(0) = 4, and f''(1) = 5. If $g(x) = f(\ln(x))$, what are g'(1) and g''(1)?

$$g'(x) = f'(\ln x)/x$$
 so $g'(1) = f'(\ln 1)/1 = f'(0) = 2$

$$g''(x) = \frac{xf''(\ln x)/x - f'(\ln x)}{x^2} = \frac{f''(\ln x) - f'(\ln x)}{x^2}$$
 so $g''(1) = \frac{4-2}{1} = 2$

(15) 5. Find the following indefinite integrals:

(a)
$$\int (x^3 + \frac{3}{x} + \cos x) \, dx$$

$$\frac{x^4}{4} + 3\ln|x| + \sin x + C$$

(b)
$$\int (2x+1)\sec^2(x^2+x) dx$$

Making the substitution $u = x^2 + x$ and du = (2x + 1) dx, we get

$$\int \sec^2 u \, du = \tan u + C = \tan(x^2 + x) + C$$

(c)
$$\int \frac{6x^2 - 4}{(x^3 - 2x + 1)^3} \, dx$$

Making the substitution $u = x^3 - 2x + 1$ and $du = (3x^2 - 2) dx$, we get

$$2\int u^{-3} du = \frac{2u^{-2}}{-2} + C = -\frac{1}{(x^3 - 2x + 1)^2} + C$$

(18) 6. Compute the following:

(a)
$$\int_{1}^{2} \frac{\sqrt{x+8}}{x} \, dx$$

$$\int_{1}^{2} (x^{-1/2} + 8x^{-1}) dx = \frac{x^{1/2}}{1/2} + 8\ln(x) \Big|_{1}^{2} = 2\sqrt{2} + 8\ln 2 - 2$$

(b) The area under the graph of $y = 2 + x^2 + \sin x$ on the interval $[0, \pi]$.

$$\int_0^{\pi} (2 + x^2 + \sin x) \, dx = 2x + \frac{x^3}{3} - \cos x \Big|_0^{\pi} = 2\pi + \frac{\pi^3}{3} - (-1) - (-1) = 2 + 2\pi + \frac{\pi^3}{3}$$

(c)
$$\int_0^\pi x^2 \sin(x^3) \, dx$$

If we make the substitution $u = x^3$ in this integral, we have $du = 3x^2 dx$ or $x^2 dx = du/3$. The integral becomes

$$\frac{1}{3} \int_0^{\pi^3} \sin(u) \, du = -\frac{1}{3} \cos(u) \Big|_0^{\pi^3} = -\frac{1}{3} \cos(\pi^3) - \left(-\frac{1}{3}\right) = \frac{1}{3} (1 - \cos(\pi^3))$$

(10) 7. In the following, A and B are constants. Let f be the function defined by

$$f(x) = \begin{cases} x^3 + Ax & \text{if } x \le 1 \\ Bx^2 + 2 & \text{if } x > 1 \end{cases}.$$

(a) What is $\lim_{x\to 1^-} f(x)$?

$$A+1$$

(b) What is $\lim_{x\to 1^+} f(x)$?

$$B+2$$

(c) How must A and B be related if f(x) is continuous at x = 1?

$$A + 1 = B + 2$$
 or $A = B + 1$.

(d) What must the values of A and B be if f(x) is differentiable at x = 1?

The derivative of $x^3 + Ax$ is $3x^2 + A$, which has the value A + 3 if x = 1. The derivative of $Bx^2 + 2$ is 2Bx, which has the value 2B at x = 1. If f(x) is differentiable at x = 1, these two values must agree. That is, we must have A + 3 = 2B. By (c), A = B + 1, so B + 4 = 2B or B = 4. From this, we get A = 5.

(9) 8. Use the linearization of $\tan x$ at $x = \pi/4$ to estimate the value of $\tan(\pi/4 + 0.13)$.

The value of $\tan x$ at $x = \pi/4$ is 1. The derivative of $\tan x$ is $\sec^2 x$, which has value 2 at $x = \pi/4$. Thus the linearization of $\tan x$ at $\pi/4$ is

$$L(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$$
 and $L\left(\frac{\pi}{4} + 0.13\right) = 1 + 2(0.13) = 1.26$.

(10) 9. Find an equation for the tangent line to the graph of $2x^3y^2 + x^2y^3 = 16$ at the point (1, 2).

Differentiating both sides of the given equation with respect to x, we get

$$2x^{3}(2y)y' + 6x^{2}y^{2} + x^{2}(3y^{2})y' + 2xy^{3} = 0 \quad \text{or} \quad y' = -\frac{6x^{2}y^{2} + 2xy^{3}}{4x^{3}y + 3x^{2}y^{2}}.$$

At the point (1,2), we have

$$y' = -\frac{24+16}{8+12} = -2,$$

and the tangent is y - 2 = -2(x - 1).

(10) 10. In this problem, assume that coordinates are given in feet. A point is moving along the x-axis in such a way that its acceleration at time t is $t + \cos 2t$ ft/sec².

(a) Suppose the velocity of the point at t = 0 is 3 ft/sec. Describe the velocity of the point as a function of t.

Let a(t) denote the acceleration. Then the velocity v(t) is an antiderivative of a(t). Thus

$$v(t) = \int (t + \cos(2t)) dt = \frac{t^2}{2} + \frac{\sin(2t)}{2} + C.$$

But 3 = v(0) = C, so

$$v(t) = \frac{t^2}{2} + \frac{\sin(2t)}{2} + 3.$$

(b) Suppose the coordinate of the point at t = 0 is 10. Describe the position of the point at time t.

The position s(t) is an antiderivative of v(t). Hence

$$s(t) = \int \left(\frac{t^2}{2} + \frac{\sin(2t)}{2} + 3\right) dt = \frac{t^3}{6} - \frac{\cos(2t)}{4} + 3t + C.$$

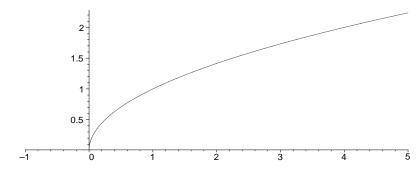
Since 10 = s(0) = -1/4 + C, we have C = 41/4 and

$$s(t) = \frac{t^3}{6} - \frac{\cos(2t)}{4} + 3t + \frac{41}{4}.$$

(8) 11. Compute the value of the Riemann sum for the function 2^x on the interval [-1, 2] using the partition -1, 0, 1, 2 and taking as the representative points the right endpoint of each subinterval.

The Riemann sum is $2^{0} \cdot 1 + 2^{1} \cdot 1 + 2^{2} \cdot 1 = 7$.

(10) 12. What point on the graph of $y = \sqrt{x}$ is closest to the point (3,0)?



A typical point on the graph is $P=(x,\sqrt{x})$, where $x\geq 0$. Let f(x) denote the square of the distance from P to (3,0). (To minimize the distance, it is enough to minimize the square of the distance.) Then

$$f(x) = (x-3)^2 + (\sqrt{x})^2 = (x-3)^2 + x = x^2 - 5x + 9.$$

Thus f'(x) = 2x - 5, which is 0 when x = 5/2. The first derivative test tells us that f(x) as a relative minimum at x = 5/2. There is one endpoint, namely x = 0, but f(0) = 9 > 2.75 = f(5/2). Thus $(5/2, \sqrt{5/2})$ is the closest point on the graph to (3,0).

(10) 13. Find equations for all horizontal and vertical asymptotes of the function $\frac{4e^{-x}+3}{7e^{-x}-2}$.

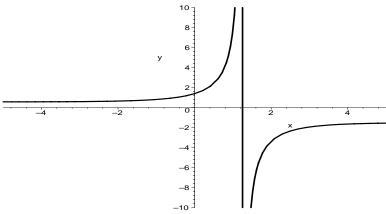
If f(x) is the given function, then

$$\lim_{x \to \infty} f(x) = \frac{4 \cdot 0 + 3}{7 \cdot 0 - 2} = -\frac{3}{2} \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{4 + 3e^x}{7 - 2e^x} = \frac{4 + 3 \cdot 0}{7 - 2 \cdot 0} = \frac{4}{7}.$$

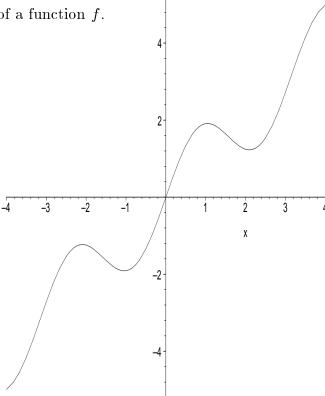
Thus there are two horizontal asymptotes, y = -3/2 and y = 4/7. The numerator of f(x) is always defined, so a vertical asymptote can only occur when the deminator is 0. If $7e^{-x} - 2 = 0$, then

$$7e^{-x} = 2$$
, $e^{-x} = 2/7$, $-x = \ln(2/7)$, or $x = -\ln(2/7) = \ln(7/2)$.

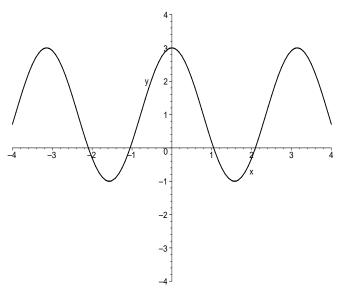
The numerator is not 0 for this value of x, so the single vertical asymptote is $x = \ln(7/2)$. Here is a graph of f.



(10) 14. Here is the graph of a function f.

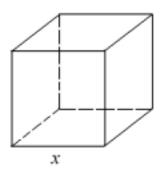


On the axes below, sketch the graph of the derivative of f.



(The original graph is actually $x + \sin 2x$ and the derivative is $1 + 2\cos 2x$.)

(10) 15. You may find it hard, but imagine you are watching a balloon in the shape of a cube being inflated. At a certain moment the volume of the balloon is 8 cubic feet and the volume is increasing at the rate of 0.3 cubic feet per minute. How fast is the surface area of the balloon increasing at that moment?



Let x be the length of an edge of the cube. Then the volume of the cube is $V = x^3$. The cube has six faces, each of which is a square with edge x. Thus the surface area of the cube is $A = 6x^2$. At the moment, V(x) = 8, that is, $x^3 = 8$ or x = 2 feet. In general,

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$
 and $\frac{dA}{dt} = 12x \frac{dx}{dt}$.

At the moment,

$$0.3 = 3 \cdot 4 \frac{dx}{dt}$$
 so $\frac{dx}{dt} = 0.025$ ft/min.

Thus

$$\frac{dA}{dt} = 12 \cdot 2(0.025) = 0.6 \text{ ft}^2/\text{min.}$$

(10) 16. In the space below, sketch the graph of a function f with the following properties: f(x) is defined and differentiable for all real numbers x except x = -3 and x = 2. The graph of f has vertical asymptotes at x = -3 and x = 2.

$$\lim_{x \to \infty} f(x) = -1$$
 and $\lim_{x \to -\infty} f(x) = 2$.

The graph of f is concave down on the intervals $(-\infty, -3)$ and $(2, \infty)$ and the graph is concave up on the interval (-3, 2).

