1. Find the following limits (5 points each), giving reasons for your answers. You may use any method from this course.

a. \[ \lim_{x \to 2} \frac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x} = -\frac{1}{8} \]

\[
\lim_{x \to 2} \frac{\sqrt{x+2} - \sqrt{2x}}{x(x-2)} = \lim_{x \to 2} \frac{(x+2) - 2x}{x(x-2)(\sqrt{x+2} + \sqrt{2x})} = \lim_{x \to 2} \frac{-2}{x(x+2)} = -\frac{1}{8}
\]

b. \[ \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x} = e^6 \]

\[ y = \left(1 + \frac{2}{x}\right)^{3x}, \quad \ln y = 3x \ln \left(1 + \frac{2}{x}\right) \]

\[ \ln y = 3 \ln \frac{1 + \frac{2}{x}}{x} = 3 \lim_{x \to \infty} \frac{\frac{2}{x}}{1 + \frac{2}{x}} = 3 \cdot \frac{2}{x^2} = \frac{6}{x^2} \]

So \[ \lim_{x \to \infty} y = e^6 \]

c. \[ \lim_{x \to 0} \frac{\sin(5x) - 5x}{x^3} = -\frac{125}{6} \]

\[ \lim_{x \to 0} \frac{5 \cos 5x - 5}{3x^2} = \lim_{x \to 0} \frac{-25 \sin 5x}{6x} = \lim_{x \to 0} \frac{-125 \cos 5x}{6} \]

\[ = -\frac{125}{6} \]
2. Find the derivatives of the following functions (7 points each). You do not need to simplify your answers.

a. If \( y = \tan(3x^2 + e) \) then \( \frac{dy}{dx} = 6x \sec^2(3x^2 + e) \)

\[
\frac{dy}{dx} = \sec^2(3x^2 + e) \cdot (6x)
\]

b. If \( y = e^{\frac{x}{x+1}} \) then \( \frac{dy}{dx} = \frac{e^{\frac{x}{x+1}}}{(x+1)^2} \)

\[
\frac{dy}{dx} = e^{\frac{x}{x+1}} \cdot \frac{d}{dx} \left( \frac{x}{x+1} \right) = e^{\frac{x}{x+1}} \cdot \frac{(x+1) \cdot 1 - x}{(x+1)^2} = e^{\frac{x}{x+1}} \cdot \left( \frac{1}{(x+1)^2} \right)
\]
3. Find the following indefinite integrals (7 points each).

a. \( \int t^2 \cos(1 - t^3) \, dt = \frac{-1}{3} \sin(1 - t^3) + C \)

\[ u = 1 - t^3 \]
\[ du = -3t^2 \, dt \]
\[ t^2 \, dt = -\frac{1}{3} \, du \]

\[ \int \cos u \left( -\frac{1}{3} \, du \right) = -\frac{1}{3} \sin u + C \]
\[ = -\frac{1}{3} \sin(1 - t^3) + C \]

b. \( \int \sqrt{x - 1} \, dx = \frac{2}{3} (x - 1)^{3/2} + C \)

\[ u = x - 1 \]
\[ du = dx \]

\[ \int \sqrt{u} \, du = \frac{u^{3/2}}{\frac{3}{2}} + C \]
\[ = \frac{2}{3} (x - 1)^{3/2} + C \]
4. Calculate the following definite integrals (7 points each).

a. \( \int_{2}^{3} \frac{\ln(x)}{x} \, dx = \frac{1}{2} \left( (\ln 3)^2 - (\ln 2)^2 \right) \)

\[ u = \ln x \]
\[ du = \frac{1}{x} \, dx \]
\[ \int_{\ln 2}^{\ln 3} u \, du = \frac{u^2}{2} \bigg|_{\ln 2}^{\ln 3} = \frac{1}{2} \left( (\ln 3)^2 - (\ln 2)^2 \right) \]

b. \( \int_{1}^{2} x \sqrt{x-1} \, dx = \frac{16}{15} \)

\[ u = x - 1 \]
\[ du = dx \]
\[ x = u + 1 \]

\[ \int_{0}^{1} (u+1)^{1/2} \, du = \int_{0}^{1} (u^{3/2} + u^{5/2}) \, du \]
\[ = \left. \frac{u^{5/2}}{\frac{5}{2}} + \frac{u^{7/2}}{\frac{7}{2}} \right|_{0}^{1} \]
\[ = \left( \frac{2}{5} + \frac{2}{3} \right) = \frac{16}{15} + \frac{10}{15} \]
\[ = \frac{26}{15} \]
5. (14 points) Estimate the area under the graph of \( f(x) = x^2 + 5x \) from \( x = 3 \) to \( x = 4 \) using 3 equally spaced approximating rectangles and right endpoints. You may leave your answer as a sum. You will receive no credit for evaluating the integral exactly.

\[
\frac{1}{3} \left( f\left( \frac{4}{3} \right) + f\left( \frac{8}{3} \right) + f\left( \frac{12}{3} \right) \right) = \frac{1}{3} \left( \frac{10}{3} \right) + 5 \left( \frac{10}{3} \right) + \frac{11}{3} \left( \frac{10}{3} \right) + 5 \left( \frac{11}{3} \right) + 5 \left( \frac{12}{3} \right) = \frac{840}{27}
\]

You don't have to multiply this out and add it up!

6. (15 points) Find the equation of the normal line to the curve described by

\[ 5x^2 y + 2y^3 = 22. \]

at the point \((2, 1)\). Any correct equation specifying this line is acceptable.\(^1\)

**Normal line:** \((y - 1) = \frac{13}{10} (x - 2)\)

Implicit diff: \[ 10xy + 5x^2 y' + 6y^2 y' = 0 \]
\[ 10\cdot 2\cdot 1 + 5\cdot 2^2 \cdot y' + 6\cdot 1^2 \cdot y' = 0 \]
\[ 20 + 20y' + 6y' = 0 \]
\[ y' = -\frac{20}{26} = -\frac{10}{13} \]

**Slope of normal line is** \(\frac{13}{10}\)

\(-1\)The normal line is perpendicular to the tangent line.
7. (14 points) A radioactive frog hops out of a pond full of nuclear waste in Oak Ridge, TN. If its level of radioactivity declines to 1/3 of the original value in 30 days, when will its level of radioactivity reach 1/100 of its original value? Note that this is an exponential decay problem.

\[
R(t) = R_0 e^{-\frac{\ln(3)}{30} t}
\]

\[
\frac{1}{3} R_0 = R_0 e^{-\frac{\ln(3)}{30} t}
\]

\[
\ln\left(\frac{1}{3}\right) = \frac{-\ln(3)}{30} t
\]

\[
t = 30 \frac{\ln(100)}{\ln(3)}
\]

8. (14 pts) Let \( f(x) = x^3 - 12x + 5 \) on the interval \([-5, 3]\). Find the absolute maximum and minimum of \( f(x) \) on this interval.

<table>
<thead>
<tr>
<th>Absolute max:</th>
<th>-21 at ( x = -2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute min:</td>
<td>-60 at ( x = -5 )</td>
</tr>
</tbody>
</table>

\[
f'(-3) = -60
\]

\[
f(-2) = 21
\]

\[
f(2) = -11
\]

\[
f(3) = -4
\]

\[
f'(x) = 3x^2 - 12 = 3(x^2 - 4)
\]

\[
f'(x) = 0 \quad x = \pm 2
\]
9. (14 points) The graph of \( y = g(x) \) is given.
   
a. For which values of \( x \) is \( g(x) \) discontinuous? Don’t worry about the endpoints at -6 and 7 in either part a or part b.
   
b. For which values of \( x \) is \( g(x) \) not differentiable?

<table>
<thead>
<tr>
<th>Discontinuous</th>
<th>( x = -3 ), ( y = -1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not differentiable</td>
<td>( y = -3, y = -2, y = -1, x = 3, x = 5 )</td>
</tr>
</tbody>
</table>

There is a vertical tangent at \( x = 3 \).
10. (14 pts) During the summer months, Terry makes and sells necklaces on the beach. Last summer, he sold the necklaces for $10 each and his sales averaged 20 per day. He also found that for each $1 increase in price sales drop by two per day. If the material for each necklace costs Terry $6, what should the selling price be to maximize his profit?

Price: $13

This is from 4.7, so there won't be a problem like this on the Spring 2009 exam.

\[
\begin{aligned}
\text{Revenue} & \quad \text{Cost} \\
P &= (10+x)(20-2x) - 6(20-2x) \\
 &= (4+x)(20-2x)
\end{aligned}
\]

\[
P'(x) = (4+x)(-2) + (1)(20-2x) = -8 - 2x + 20 - 2x = -4x
\]

\[
P'(x) = 0 \quad \text{when} \quad x = 3
\]

For \( x < 3 \), \( P'(x) > 0 \) and for \( x > 3 \), \( P'(x) < 0 \) so \( P(x) \) has max at \( x = 3 \)

Selling price should be $13
11. (15 points) An open cylindrical can (without top) is to be constructed to hold $16\pi$ cubic cm of liquid. The cost of the material for the bottom is $2$ per cm$^2$, and the cost of the material for the curved surface is $1$ per cm$^2$. Find the radius and the height of the least expensive can. (The area of the curved surface is the circumference of the circle times the height.)

\[
C = 2\pi r^2 + 2\pi rh
\]
\[
V = 16\pi = \pi r^2 h \Rightarrow h = \frac{16}{r^2}
\]

\[
C = 2\pi r^2 + 2\pi r\left(\frac{16}{r^2}\right) = 2\pi r^2 + 2\pi \left(\frac{16}{r}\right) = 2\pi \left[r^2 + \frac{16}{r}\right]
\]

\[
C' = 2\pi \left[2r - \frac{16}{r^2}\right]
\]

\[
C'(r) = 0 \text{ when } 2r = \frac{16}{r^2} \Rightarrow r^3 = 8 \Rightarrow r = 2
\]

So $r = 2$, $h = 4$
12. (15 points) Use linear approximation or differentials to find an approximate value for \( \sqrt{8.5} \).

Approx value: \[ \frac{49}{24} \]

\[
\begin{align*}
\sqrt{8.5} & \approx \sqrt{8} + \frac{1}{2} (8)^{-\frac{1}{2}} (8.5 - 8) \\
& = 2 + \frac{1}{2} (\frac{1}{8}) (.5) \\
& = 2 + 0.0625 \\
& = \frac{49}{24}
\end{align*}
\]

13. (15 points) The altitude of a triangle is increasing at a rate of one ft/min while the area is increasing at a rate of 2 ft\(^2\)/min. At what rate is the base of the triangle changing when the altitude is 10 ft and the area is 100 ft\(^2\)?

Rate of change: \[ -\frac{8}{5} \]
14. (15 points) Sketch the graph of the function \( f(x) = \frac{24}{x^3 + 8} \). For this function,

\[
f'(x) = \frac{-72x^2}{(x^3 + 8)^2} \quad \text{and} \quad f''(x) = \frac{288x(x^3 - 4)}{(x^3 + 8)^3}.
\]

<table>
<thead>
<tr>
<th>Horizontal asymptote(s):</th>
<th>( y = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical asymptote(s):</td>
<td>( x = -2 )</td>
</tr>
<tr>
<td>Increasing:</td>
<td>Nowhere</td>
</tr>
<tr>
<td>Decreasing:</td>
<td>((-\infty, -2) \cup (-2, \infty))</td>
</tr>
<tr>
<td>Concave up:</td>
<td>((-2, 0) \cup (4^{1/3}, \infty))</td>
</tr>
<tr>
<td>Concave down:</td>
<td>((-\infty, -2) \cup (0, 4^{1/3}))</td>
</tr>
<tr>
<td>Relative max/min:</td>
<td>None</td>
</tr>
<tr>
<td>Inflections:</td>
<td>( x = 4^{1/3}, x = 0 )</td>
</tr>
</tbody>
</table>

\[
\lim_{x \to \pm \infty} f(x) = 0
\]

\[
\begin{array}{c|c|c}
\hline
& \pm \infty & 0 \\
\hline
\pm \infty & \uparrow & \downarrow \\
0 & \downarrow & \uparrow \\
4^{1/3} & \uparrow & \downarrow \\
\hline
\end{array}
\]

\( x = -2 \) is a vertical asymptote so it can't be an inflection.