Old business. The general power rule can be derived from the calculus of the exponential and logarithm functions. This will explain why the exponent is decreased by 1 in the power rule with constant exponent while it remains unchanged for the exponential function with constant base.

If \( f(x) = x^n \), we may write \( f(x) = e^{n \ln x} \). Then

\[
f'(x) = e^{n \ln x} \frac{d}{dx}(n \ln x) = x^n \frac{n}{x} = nx^{n-1}.
\]

Consider this graph.

![Graph of trigonometric functions](image)

Trigonometry. The functions \( \sin t \) and \( \cos t \) describe the process of drawing a circle. Here \( \cos t \) is the \( x \)-coordinate and \( \sin t \) the \( y \) coordinate of a point on the unit circle with equation \( x^2 + y^2 = 1 \), where \( t \) is supposed to be an intrinsic quantity known as angle that describes the location of the point on the circle.

One way to measure angles is to use the length of the arc on the circle. Another way is to use the area of the sector formed by the arc and the radii to the endpoints of the arc. Unfortunately, both of these approaches require calculus that has not yet been described in this course. Another approach is based on the idea of rotation as a rigid motion. Again, the details are missing in what has been done in this course. The results are true and can be proved. but our description of the functions will rely on an intuition about angles.

Periodicity. When you draw a circle, you eventually get back to your starting point. Since continued drawing gives no new points on the curve, one usually stops at this point. However, nothing forces you to stop, and you could continue indefinitely wrapping a number line around the circle. The point where you first meet your starting point defines a period of the functions describing the coordinates of the point on the circle. Although described initially only for posi-
tive value of the angle, it is easy to use periodicity to
discover how negative angles should be interpreted.
The first slide began with a graph showing two periods
of $y = \sin(x)$.

Since many graphing programs give independent scal-
ing of the axes, all graphs of $y = A \sin bx$ would look
the same. Moreover, the graphs of $y = \sin x$ and
$y = \cos x$ differ only by a horizontal translation.

A change of unit for angular measurement also has the
effect of changing the scale on the $x$ axis, or equiva-
ently multiplying $x$ by a constant before evaluating
the function. As with exponentials and logarithms,
several scales are in common use (most calculators
allow three choices), but calculus selects a natural
measure that leads to the best differentiation formulas.

The graph shows scales on the horizontal and ver-
tical axes that play different roles: every horizontal
distance can be measured with respect to the period
of the function (the smallest positive number $p$ such
$f(x + p) = f(x)$); and vertical distances are mea-
sured with respect to the amplitude (the distance from
the central value to the maxima or minima), which is
1 for $\sin x$.

**Calculus.** The graph of $y = \sin x$ looks smooth,
suggesting that this function is differentiable. Investi-
gating the possibility of computing the derivative of
$\sin x$ at $x = 0$ already reveals the need for

$$\lim_{x \to 0} \frac{\sin x}{x}.$$  

Since a change of angular measurement is equivalent
to replacing $\sin x$ by $\sin bx$ for some $b$, and

$$\lim_{x \to 0} \frac{\sin bx}{x} = \lim_{x \to 0} b \frac{\sin bx}{bx} = b \lim_{x \to 0} \frac{\sin x}{x},$$

the value of this limit depends on the scale of mea-
urement, though the existence of the limit does not.

Once this limit is shown to exist, enough other limits
can be calculated to show that $\sin x$ is differentiable
everywhere. First, consider

$$\frac{1 - \cos x}{x} = \frac{1 - \cos^2 x}{x(1 + \cos x)}$$

$$= \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= (\sin x) \frac{\sin x}{x} \frac{1}{1 + \cos x}.$$ 

Each factor in the last expression has a limit, and one of them has limit zero, so we have

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$ 

To differentiate $\sin x$, start with the addition formula for this function

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$ 

Subtracting $\sin x$ and grouping terms gives

$$\sin(x + h) - \sin x = \sin x (\cos h - 1) + \cos x \sin h.$$ 

Dividing by $h$ and selecting the most useful of distributing this factor gives

$$\frac{\sin(x + h) - \sin x}{h} = \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}.$$ 

Each piece has a limit as $h \to 0$, so

$$\frac{d}{dx} (\sin x) = \left( \lim_{h \to 0} \frac{\sin h}{h} \right) \cos x.$$ 

The best unit of measurement is one for which

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$ 

Such a unit is one that measures angles by the length of the arc on the unit circle, so the measure of once around the unit circle is $2\pi$. This unit of measurement is called radian measure. When using your calculator for exercises in this course, you must select that mode. All trigonometric functions have period $2\pi$. 

lec18ch.12s.1,2,3.5  
lec18ch.12s.1,2,3.6
Radian measure of important angles. Since the whole circle is $2\pi$ radians, a semicircle, which is the measure of a straight angle, is $\pi$ radians. Half of a straight angle is a right angle whose measure is thus $\pi/2$. The sum of the interior angles of a triangle is a straight angle, so an isosceles right triangle has one right angle of measure $\pi/2$ and two acute angles, each of size $\pi/4$. The equilateral triangle has three equal angles, each of size $\pi/3$. If you bisect one of the angles to get an angle of $\pi/6$, that angle bisector must be perpendicular to the opposite side, and it divided the original triangle into congruent parts, so the angle bisector is also a median and a perpendicular bisector of the opposite side. These examples give some angles whose trigonometric functions can be found exactly.

The above description required only a little bit of Euclid to give a description that used only radian measure. Since the degree measure of these angles are also known, any of these values can be used to relate the two measures. In particular, the straight angle of $\pi$ radians is $180^\circ$.

What does “co” mean? Trigonometric functions occur in pairs: sine and cosine; tangent and cotangent; secant and cosecant. For acute angles, these are ratios of sides of a right triangle, and “co” indicates the ratio that would be obtained by using a particular definition for the other acute angle of the triangle. Since the sum of the acute angles of a right triangle is $\pi/2$, this “other” angle is the complement, and the angle is $\pi/2 - x$. In particular,

$$\cos x = \sin \left( \frac{\pi}{2} - x \right),$$

and

$$\frac{d}{dx} \cos x = \frac{d}{dx} \left( \sin \left( \frac{\pi}{2} - x \right) \right)$$

$$= \cos \left( \frac{\pi}{2} - x \right) \cdot \frac{d}{dx} \left( \frac{\pi}{2} - x \right)$$

$$= -\cos \left( \frac{\pi}{2} - x \right)$$

$$= -\sin x.$$
differentiated directly are
\[
\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}.
\]

The quotient rule allows these to be differentiated now that we know the derivatives of \(\sin x\) and \(\cos x\). The results are:

\[
\frac{d}{dx}(\tan x) = \sec^2 x
\]

\[
\frac{d}{dx}(\sec x) = \sec x \tan x
\]

All that remains is to get some practice using these new formulas.