Section 3.3 The chain rule. The chain rule tells us that if the derivatives of functions \( f \) and \( g \) are known, then the composition \( f \circ g \) has a derivative, and there is a formula that gives it in terms of \( f \) and \( g \) and their derivatives. In this notation, the formula looks awkward, so it is better not to write it until we have seen, and interpreted a more natural form of the rule.

The linear case. First, let us see how composition of linear functions behaves. Let \( f(x) = ax + b \) and \( g(x) = cx + d \), and introduce \( H(x) = (f \circ g)(x) \). Then \( H(x) = (f \circ g)(x) = f(g(x)) \), so we must replace the \( x \) used in the definition of \( f \) with the expression \( g(x) \) that contains a variable \( x \) that is playing a different role. Poor \( x \) is overworked, and unnecessarily so, since its role in these formulas is just to illustrate how the functions are computed. We could equally well have defined the function \( f \) by writing \( f(y) = ay + b \). To find \( (f \circ g)(x) \) we need only substitute \( y = g(x) = cx + d \) for the \( y \) in this expression. The result is

\[
(f \circ g)(x) = a(cx + d) + b = acx + (ad + b).
\]

One may be struck by how complicated the constant term is, but what should be noted is:

If the derivatives of \( f \) and \( g \) are constant, then the derivative of the composition \( f \circ g \) is the product of those constants.

This statement is deliberately written in words in order to avoid writing a misleading formula. Mathematical statements have both a hypothesis and a conclusion, and should be used by first verifying the hypothesis to justify using the conclusion. Excessive use of formulas for the conclusions of theorems makes the application of the theorem seem like little more the substitution of one expression for another. This is only a simple finish to the process that built the structure that made this possible.

In applied calculus, the structure is the important part, not the formulas with which it is decorated.

General formulation. Working with the composition of functions is simplified by introducing different names for the elements in the spaces connected by the functions. The picture associated with \( H = f \circ g \) is

\[
\begin{align*}
H & : \{x\} \xrightarrow{g} \{y\} \xrightarrow{f} \{z\}.
\end{align*}
\]

(A similar picture was used in lecture 2, but it emphasized the spaces connected by the functions as part of a study of the domain and range of a function.)

When we want to do Calculus, we need to analyze the relation between the composite function \( f \circ g \) and its parts \( f \) and \( g \). What our current picture says is that we have

\[
\begin{align*}
y &= g(x) \\
z &= f(y) \\
\end{align*}
\]

combine to give

\[
z = H(x).
\]

To study this at \( x = a \), we need to produce \( b = g(a) \) to get the corresponding value of the variable \( y \) that is used to describe the function \( f \). Then, \( c = f(b) \) is the value of the variable \( z \) giving the value in the range of \( H = f \circ g \) corresponding to \( x = a \) in the domain.

The main claim of differential calculus is that the behavior of the graph of the function near a particular point can be approximated by behavior of the tangent line at that point. In this example we have three different functions whose graphs live in three different spaces. The function \( g \) is graphed in an \((x, y)\) plane in which we have marked the point \((a, b)\) on the graph; the function \( f \) is graphed in a \((y, z)\) plane in which we have marked the point \((b, c)\) on the graph; and the function \( H \) is graphed in an \((x, z)\) plane in which we have marked the point \((a, c)\) on the graph. The tangent lines to these curves at the marked points are

\[
\begin{align*}
(y - b) &= g'(a)(x - a) \quad (T_g) \\
(z - c) &= f'(b)(y - b) \quad (T_f) \\
(z - c) &= H'(a)(x - a) \quad (T_H)
\end{align*}
\]

For the linear approximation to \( H \) to be built from the linear approximations to \( f \) and \( g \), we would have \( H'(a) = f'(b)g'(a) \). Note that \( f' \) is evaluated at a different point that the other functions, but that is necessary because \( f \) has a different domain than the
other functions. When expressed as functions of $x$, 

\[ H'(x) = f'(g(x)) \cdot g'(x). \]

This is a (correct) statement of the (true) chain rule.

**A word about the proof.** It can almost be proved by writing the difference quotient 

\[ \frac{H(x + h) - H(x)}{h} \]

as 

\[ \frac{f(g(x + h)) - f(g(x))}{g(x + h) - g(x)} \cdot \frac{g(x + h) - g(x)}{h} \]

and taking limits. This breaks down if it is possible for $g(x + h) = g(x)$ for small $h \neq 0$. A correct proof requires a technical device to avoid this. The modified proof is reasonably straightforward, but it undermines the importance given to limits in calculus textbooks, so it is often hidden.

For the moment, this will be a single new complicated name for a variable, and calculus gives an expression for it in terms of $x$.

If, also, $z$ can be expressed in terms of $y$, there will be a single new complicated name for a variable 

\[ \frac{dz}{dy} \]

that can be expressed in terms of $y$. These pieces allow $z$ to be expressed in terms of $x$ and lead to the introduction of 

\[ \frac{dz}{dx}. \]

In this notation, the chain rule says that 

\[ \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}, \]

which looks obvious, and should be easy to remember. It is not as obvious as it looks, since it is still the same chain rule described above. The expressions
like $dy/dx$ are not fractions, since this is just a complicated name for a derivative, and things like $dx$ and $dy$ that appear to be part of the expression have not (yet) been given an independent meaning. Furthermore, the factor $dz/dy$ is defined so that it is found as an expression in terms of $y$, but everything else is supposed to be expressed in terms of $x$. However, we have an expression for $y$ in terms of $x$ that can be used in interpreting this factor.

The description has gone on too long. It is time for some examples.

**Multiplicative inverses and the quotient rule.** Let $z = 1/y = y^{-1}$ and $y = q(x)$. The general power rule gives

$$
\frac{dz}{dy} = (-1)y^{-2} = -1/y^2,
$$

and $dy/dx = q'(x)$. Expressing everything in terms of $x$.

$$
\frac{dz}{dy} = \frac{-1}{y^2} = \frac{-1}{q(x)^2},
$$

and

$$
\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{-1}{q(x)^2} \cdot q'(x) = \frac{-q'(x)}{q(x)^2}.
$$

Thus, if

$$
f(x) = \frac{p(x)}{q(x)} = p(x) \cdot \frac{1}{q(x)},
$$

then

$$
f'(x) = p(x) \cdot \frac{-q'(x)}{q(x)^2} + p'(x) \cdot \frac{1}{q(x)} = \frac{q(x)p'(x) - p(x)q'(x)}{q(x)^2}.
$$