COMBINATORIAL AND CYCLOTOMIC PROPERTIES OF CERTAIN TRIDIAGONAL MATRICES

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Abstract. Under consideration are the properties of the determinants and permanents of a certain class of symmetric tridiagonal matrices whose off-diagonal elements are in geometric progression. When the matrix is infinite these functions are power series in the ratio of the progression with the unit circle as a natural boundary. The coefficients have combinatorial significance since they are the number of partitions into parts restricted in various ways, including the restricted partitions of the Rogers-Ramanujan identities. When the matrix is finite a study is made of the behavior of the determinant and permanent at roots of unity as the order of the matrix tends to infinity.

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1. Introduction and Notation. Associated with the geometric progression of real or complex numbers

$$a, ar, ar^2, ar^3, \cdots$$

we define a tridiagonal $n \times n$ matrix by

$$M_n = M_n(a,r) = \begin{vmatrix} 1 & a & 0 & 0 & \cdots 0 & 0 \\ a & 1 & ar & 0 & \cdots 0 & 0 \\ 0 & ar & 1 & ar^2 & \cdots 0 & 0 \\ 0 & 0 & ar^2 & 1 & \cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots 1 & ar^{n-2} \\ 0 & 0 & & & \cdots ar^{n-2} & 1 \end{vmatrix}$$

If we expand the determinant of M_n according to minors of elements of the last row, we have

(1) $$\det M_n = \det M_{n-1} - a^2 r^{2n-4} \det M_{n-2}.$$

For the permanent of M_n we have

(2) $$\text{per } M_n = \text{per } M_{n-1} + a^2 r^{2n-4} \text{ per } M_{n-2}$$

with the initial conditions

$$\det M_0 = \text{per } M_0 = \det M_1 = \text{per } M_1 = 1.$$

It follows at once that

$$\text{per } M_n(a,r) = \det M_n(ai,r)$$

and that both functions are polynomials in a^2 and r^2. In fact, a little inductive reasoning, using (1) and (2), shows that both functions are of degree $\left[\frac{n}{2}\right]$ in a^2 and of degree $\left[\frac{n-1}{2}\right]\left[\frac{n}{2}\right]$ in r^2.

In case $|r| < 1$ we can consider

$$\det M(a,r) \quad \text{and} \quad \text{per } M(a,r)$$

where

$$M(a,r) = \lim_{n\to\infty} M_n(a,r).$$

These functions will be double power series in a^2 and r^2 with integer coefficients. (Actually, these coefficients and the corresponding formal power series exist whether $|r| < 1$ or not).

For us, the most interesting matrices $M_n(a,r)$ are those in which a is an integer power of r, say $a = r^s$. We now have polynomials in a single variable r^2 with a parameter s. Calling $r^2 = x$ we write

$$D_n(s|x) = \det M_n(r^s,r)$$

$$\Delta_n(s|x) = \text{per } M_n(r^s,r)$$

and

$$D(s|x) = \lim_{n\to\infty} D_n(s|x)$$

$$\Delta(s|x) = \lim_{n\to\infty} \Delta_n(s|x).$$

Accordingly (1) and (2) become

(3) $$D_n(s|x) = D_{n-1}(s|x) - x^{s+n-2}D_{n-2}(s|x)$$

(4) $$\Delta_n(s|x) = \Delta_{n-1}(s|x) + x^{s+n-2}\Delta_{n-2}(s|x)$$

with

(5) $$D_0(s|x) = \Delta_0(s|x) = D_1(s|x) = \Delta_1(s|x) = 1.$$

Applying (3) and (4) a few times gives us

$$D_2(s|x) = 1-x^s$$
$$D_3(s|x) = 1-x^s-x^{s+1}$$
$$D_4(s|x) = 1-x^s-x^{s+1}-x^{s+2}+x^{2s+2}$$
$$D_5(s|x) = 1-x^s-x^{s+1}-x^{s+2}-x^{s+3}+x^{2s+2}+x^{2s+3}+x^{2s+4}$$

....

$$\Delta_2(s|x) = 1+x^s$$
$$\Delta_3(s|x) = 1+x^s+x^{s+1}$$
$$\Delta_4(s|x) = 1+x^s+x^{s+1}+x^{s+2}+x^{2s+2}$$
$$\Delta_5(s|x) = 1+x^s+x^{s+1}+x^{s+2}+x^{s+3}+x^{2s+2}+x^{2s+3}+x^{2s+4}$$

....

Of course for small s some of the above terms have the same power of x and should be collected under one coefficient. Thus when $s = 1$

$$x^{s+3} = x^{2s+2} = x^4$$

and so $D_5(s|x)$ has no term in x^4.

The degree of D_n and Δ_n is seen to be $s\left[\frac{n}{2}\right] + \left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]$.

2. Connection with Continued Fractions. The D's and Δ's are related to the convergents of the two continued fractions of Ramanujan type. Let

$$f = 1 + \frac{-x^s|}{|1} + \frac{-x^{s+1}|}{|1} + \frac{-x^{s+2}|}{|1} + \cdots$$

$$g = 1 + \frac{x^s|}{|1} + \frac{x^{s+1}|}{|1} + \frac{x^{s+2}|}{|1} + \cdots$$

Then the n^{th} convergents of f and of g are respectively

$$D_{n+1}(s|x)/D_n(s+1|x)$$

and

$$\Delta_{n+1}(s|x)/\Delta_n(s+1|x).$$

Thus for $n = 3$

$$1 + \frac{x^s|}{|1} + \frac{x^{s+1}|}{|1} + \frac{x^{s+2}|}{|1} = \frac{1+x^s+x^{s+1}+x^{s+2}+x^{2s+2}}{1+x^{s+1}+x^{s+2}} = \frac{\Delta_4(s|x)}{\Delta_3(x+1|x)}.$$

For the general n one has only to use (3) and (4) in an induction proof.

3. Generating Functions. We introduce the two generating functions

$$F(s,x,z) = \sum_{n=0}^{\infty} D_n(s|x)z^n$$

$$\Phi(s,x,z) = \sum_{n=0}^{\infty} \Delta_n(s|x)z^n.$$

In view of the recurrences (3) and (4) we see that F and Φ satisfy the functional equations

(6) $$(1-z)F(s,x,z) = 1 - x^s z^2 F(s,x,xz)$$

(7) $$(1-z)\Phi(s,x,z) = 1 + x^s z^2 \Phi(s,x,xz).$$

If we multiply and divide the last terms on the right by $(1-xz)$ and reapp the functional equations (6) and (7) we get

$$(1-z)F(s,x,z) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{m(m+s-1)} z^{2m}}{(1-xz)(1-x^2z)\cdots(1-x^mz)}$$

$$(1-z)\Phi(s,x,z) = \sum_{m=0}^{\infty} \frac{x^{m(m+s-1)} z^{2m}}{(1-xz)(1-x^2z)\cdots(1-x^mz)}.$$

On the other hand

$$(1-z)F(s,x,z) = D_0 + \{D_1-D_0\}z + \{D_2-D_1\}z^2 + \cdots$$

Hence

$$D(s|x) = \lim_{n\to\infty} D_n(s|x) = \lim_{z\to 1} (1-z)F(s,x,z).$$

That is

(8) $$D(s|x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{m(m+s-1)}}{(1-x)(1-x^2)\cdots(1-x^m)}.$$

In the same way we have

(9) $$\Delta(s|x) = \sum_{m=0}^{\infty} \frac{x^{m(m+s-1)}}{(1-x)(1-x^2)\cdots(1-x^m)}.$$

4. Combinatorial Interpretation of D and Δ. If we properly collect the terms in the expansion of $D(s|x)$ and $\Delta(s|x)$ in powers of x so that

$$D(s|x) = \sum_{n=0}^{\infty} d_n(s)x^n$$

$$\Delta(s|x) = \sum_{n=0}^{\infty} \delta_n(s)x^n$$

then it is clear that the coefficients d and δ are integers and, by (9), the δ's are $\geqslant 0$. Their meaning is explained by the following theorem.

Theorem 1. Let $C_n(s)$ be the set of all partitions of n in which each part exceeds the number of parts by at least $s - 1$. Let $e_n(s)$ be the number of such partitions having an even number of parts and let $\omega_n(s)$ be the number of partitions into an odd number of parts. Then

$$d_n(s) = e_n(s) - \omega_n(s)$$

$$\delta_n(s) = e_n(s) + \omega_n(s).$$

Thus $\delta_n(s)$ is the cardinality of $C_n(s)$.

Proof. Let m be any positive integer. Consider the subset σ_m of $C_n(s)$ comprised of those partitions having precisely m parts. This subset will be empty for $n < m(m+s-1)$ since each of the m parts must be not less than $m + s - 1$. The Ferrer's graph of any member of σ_m will therefore contain a rectangle of m rows and the first $(m+s-1)$ columns. Removing these nodes, we are left with the graph of a partition of $n - m(m+s-1)$ into parts not exceeding m. That is, the cardinality of σ_m is the coefficient of x^n in the expansion of

$$\frac{x^{m(m+s-1)}}{(1-x)(1-x^2)\cdots(1-x^m)}$$

in powers of x. Summing this generator over all m with or without alternating signs and recalling (8) and (9) gives the theorem.

This theorem remains true if $C_n(s)$ is replaced by the set of all partitions of n whose largest part P occurs at least $P + s - 1$ times and $e_n(s)$ now enumerates those partitions whose largest part is even. This is seen at once by reading the Ferrer's graph by columns instead of rows.

5. Examples. If we take s to be zero, (3) and (4) become

$$D_n(0|x) = D_{n-1}(0|x) - x^{n-2}D_{n-2}(0|x)$$

$$\Delta_n(0|x) = \Delta_{n-1}(0|x) + x^{n-2}\Delta_{n-2}(0|x).$$

If we apply these recurrences repeatedly until the first dozen coefficients remain unaffected by further iterations we find

$$D(0|x) = 0-x+x^4+x^5+x^6+x^7-x^{11}-x^{12}+\cdots$$

$$\Delta(0|x) = 2+x+2x^2+2x^3+3x^4+3x^5+5x^6+5x^7+7x^8+8x^9+10x^{10}+11x^{11}+15x^{12}+\cdots$$

For $n = 10$, for example, the partitions in $C_{10}(0)$, namely those partitions of 10 whose parts are greater than or equal to the number of parts minus 1, are seen to be

10	5 + 5	3 + 3 + 4
	4 + 6	2 + 4 + 4
	3 + 7	2 + 3 + 5
	2 + 8	2 + 2 + 6
	1 + 9	

There are 10 of these, 5 with an even number of parts. Hence the term $10x^{10}$ in $\Delta(0|x)$ and the absence of a term in x^{10} in $D(0|x)$.

If now we set $s = 1$ we are considering in $C_n(1)$, partitions of n in which the parts are at least as large as the number of parts, so that our rectangle is now a square (the Durfee square of the graph). The generators (8) and (9) become

$$D(1|x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{m^2}}{(1-x)\cdots(1-x^m)} = 1-x-x^2-x^3+x^6+x^7+2x^8+\cdots$$

$$\Delta(1|x) = \sum_{m=0}^{\infty} \frac{x^{m^2}}{(1-x)\cdots(1-x^m)} = 1+x+x^2+x^3+2x^4+2x^5+3x^6+\cdots$$

But by the first Rogers-Ramanujan identity [1,p.288]

$$\sum_{m=0}^{\infty} \frac{x^{m^2}}{(1-x)\cdots(1-x^m)} = \prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+1})(1-x^{5n+4})}.$$

Hence $\Delta(1,x)$ also generates the number of partitions into parts congruent to 1 or 4 modulo 5. $\Delta(1,x)$ is the generator of still another kind of partition namely partitions into parts differing by 2 or more, [1, p.289].

For $s = 2$ we are considering partitions whose parts exceed the number of parts by 1 or more. The generators are

$$D(2|x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{m(m+1)}}{(1-x)\cdots(1-x^m)} = 1-x^2-x^4+x^7+x^8+\cdots$$

$$\Delta(2|x) = \sum_{m=0}^{\infty} \frac{x^{m(m+1)}}{(1-x)\cdots(1-x^m)} = 1+x^2+x^3+x^4+x^5+x^6+2x^7+2x^8+\cdots.$$

By the second Rogers-Ramanujan identity [1,p.288]

$$\Delta(2|x) = \prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+2})(1-x^{5n+3})} .$$

Hence $\Delta(2|x)$ generates partitions into parts congruent to 2 or 3 modulo 5. It likewise generates partitions into parts > 1 and differing by 2 or more.

6. A Non-existence Theorem. We have just seen in the two examples of $s = 1$ and $s = 2$ that $\Delta(s|x)$ generates partitions into parts taken from a particular set of numbers. We can show that there are no further examples of this phenomenon.

Theorem 2. Let $s > 2$. Then $\Delta(s|x)$ generates the number of partitions into parts taken from no set of integers whatever.

Proof. We shall need the support of

Lemma 1. Denote by $p_k(n)$ the number of partitions of n into not more than k parts. Then

$$p_1(n) = 1$$

$$p_2(n) = 1 + \left[\frac{n}{2}\right]$$

$$p_3(n) = \left[\frac{n^2+6n+15}{12}\right].$$

These results are well known (see [2] p. 146).

Now suppose, if possible, there is a set of integers

$$a_1 < a_2 < a_3 < \cdots$$

finite or infinite such that

$$\prod_{i=1} (1-x^{a_i})^{-1} = \Delta(s|x) = \sum_{m=0}^{\infty} \frac{x^{m(m+s-1)}}{(1-x)(1-x^2)\cdots(1-x^m)}$$

(10)

$$= 1 + \frac{x^s}{1-x} + \frac{x^{2s+2}}{(1-x)(1-x^2)} + \frac{x^{3s+6}}{(1-x)(1-x^2)(1-x^3)} +\cdots.$$

We shall show that the coefficient of x^{4s+3} in the expansion of the product on the left exceeds that on the right, a contradiction. If we identify coefficients of x^n for $n = 1,2,3,\ldots,2s-1$ we see that

$$a_i = s + i - 1 \qquad i = 1(1)s.$$

Since

$$\frac{x^{2s+2}}{(1-x)(1-x^2)} = \sum_{n=0}^{\infty} p_2(n)x^{2s+n+2}$$

the coefficients of x^n vanish on the right for $n = 2s(1)3s$ and we find

$$a_{s+i} = 3s + i \qquad (i=1(1)s).$$

Thus the first $2s$ parts a_i are

(11) $s, s+1, s+2, \ldots, 2s-1;\ 3s+1, 3s+2, 3s+3, \ldots, 4s.$

Now consider the number of partitions of $4s+3$ into parts taken from the above set. These can be enumerated systematically by considering, in turn, the number of parts in each such partition. There are precisely 3 partitions into 2 parts namely

$$s + (3s+3), \quad (s+1) + (3s+2), \quad (s+2) + (3s+1).$$

A 3-part partition is of the form

$$(s-1+i) + (s-1+j) + (s-1+k) = 4s + 3$$

which is equivalent to

$$i + j + k = s + 6 \qquad 1 \leqslant i \leqslant j \leqslant k.$$

From the total number of these partitions, namely

$$p_3(s+6) - p_2(s+6)$$

we must subtract those having $k > s$ of which there are

$$p_2(2) + p_2(3) + p_2(4) + p_2(5) - 4 = 6$$

corresponding to $k = s+4$, $s+3$, $s+2$, and $s+1$ respectively. In all then, the 3-part partitions of $4s + 3$ account for

$$p_3(s+6) - p_2(s+6) - 6 = \left[\frac{s^2+18s}{12}\right] - \left[\frac{s}{2}\right] - 3.$$

The 4-part partitions of $4s + 3$ are equivalent to partitions of 7 into four parts $\leqslant s$. There are

$$1 + 1 + 1 + 4$$

$$1 + 1 + 2 + 3$$

$$1 + 2 + 2 + 2$$

so their number is 2 or 3 according as $s = 3$ or $s > 3$.

When $s = 3$ there is one 5-part partition of $4s + 3 = 15$, namely

$$3 + 3 + 3 + 3 + 3$$

and none in case $s > 3$ which makes up for the missing 4-part partition when $s = 3$.

Putting all the cases together we get

$$c = \left[\frac{s^2+18s}{12}\right] - \left[\frac{s}{2}\right] + 3$$

for the number of partitions of $4s + 3$ into parts taken from (11).

Now the coefficient of x^{4s+3} on the right of (10) is clearly

$$c' = p_1(3s+3) + p_2(2s+1) + p_3(s-3) = 2 + s + \left[\frac{s^2+6}{12}\right].$$

Comparing this with c we find

$$c - c' = \left[\frac{s^2}{12} + \frac{s}{2}\right] - \left[\frac{s^2}{12} + \frac{1}{2}\right] - \left[\frac{s}{2}\right] + 1.$$

Now by separating a few cases of s and recalling that no square is congruent to 2 modulo 3 we can easily show that the first three terms of this sum destroy each other. Thus

$$c = c' + 1$$

is already too large; introducing more parts a_i in the partition of $4s + 3$ will not decrease the coefficient of x^{4s+3} in the expansion of the left side of (10). Hence we have reached a contradiction.

7. Computational Efficiency. Given s, the problem of computing a large table of the coefficients $d_n(s)$ and $\delta_n(s)$ of D and Δ is most efficie[ntly] programmed via the recurrences (3) and (4) and not by expanding (8) or (9). The first thousand coefficients can be obtained in about 10 seconds on a large computer; the larger the s the cheaper the computation. The results of such computations including the asymptotic behavior of $\delta_n(s)$ will be given in another paper.

[8.] D_n and Δ_n on the Unit Circle. We now consider the polynomials $D_n(s|x)$ and $\Delta_n(s|x)$ when x is a root of unity. For this we need some further notation. We generalize (3) and (4) by considering the difference equation.

$$w_n = w_{n-1} + qx^{s+n-2}w_{n-2} \qquad (w_0=w_1=1) \tag{12}$$

where q is a parameter and

$$w_n = w_n(s|x,q).$$

If we introduce the generating function

$$G(z) = G(s|x,q,z) = \sum_{n=0}^{\infty} w_n(s|x,q)z^n$$

then in view of (12) we get

$$(1-z)G(z) = 1 + qx^s z^2 G(xz)$$

as a generalization of (6) and (7). Iterating this functional equation k times we obtain

$$(1-z)G(z) = \sum_{m=0}^{k-1} \frac{q^m x^{m(m+s-1)} z^{2m}}{(1-xz)(1-x^2z)\cdots(1-x^m z)} + \frac{q^k x^{k(k+s-1)} z^{2k} G(x^k z)}{(1-xz)(1-x^2z)\cdots(1-x^{k-1}z)}. \tag{13}$$

We also introduce two recurring sequences F_n and F_n'. F_n is the standard Fibonacci sequence

$$F_0 = 0 \qquad F_1 = 1 \qquad F_n = F_{n-1} + F_{n-2}$$

while

$$F'_0 = 0 \qquad F'_1 = 1 \qquad F'_n = F'_{n-1} - F'_{n-2}.$$

As is well known

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \sim \alpha^n/\sqrt{5} \quad \text{as} \quad n \to \infty$$

where

$$\alpha = (1+\sqrt{5})/2, \qquad \beta = (1-\sqrt{5})/2$$

while

$$F'_n = \frac{\sqrt{12}}{3} \sin (\pi n/3)$$

is a periodic function of n of period 6 beginning

$$0,1,1,0,-1,-1,0,1,\ldots$$

Finally we let θ be any primitive kth root of unity. To answ the question about the behavior of $D_n(s|x)$ and $\Delta_n(s|x)$ as $n \to \infty$ and $x = \theta$ we have

<u>Theorem 3.</u> As $n \to \infty$, $|\Delta_n(s|\theta)|$ tends to infinity like $\alpha^{n/k}$. If k is even $|D_n(s|\theta)|$ behaves in the same way. If k is odd $|D_n(s|\theta)|$ is a periodic function of n of period $6k$.

<u>Proof</u>. If $k = 1$ or $k = 2$ the proof is easy. For $k = 1$, $\theta = 1$ and (3) and (4) become

$$D_n(s|1) = D_{n-1}(s|1) - D_{n-2}(s|1) \qquad (D_0 = D_1 = 1)$$

$$\Delta_n(s|1) = \Delta_{n-1}(s|1) + \Delta_n(s|1) \qquad (\Delta_0 = \Delta_1 = 1).$$

It follows that there are functions of n alone and that

$$D_n(s|1) = F'_{n+1}, \qquad \Delta_n(s|1) = F_{n+1}.$$

From this the theorem follows for $k = 1$.

For $k = 2$, $\theta = -1$ and (3) and (4) become

$$D_n(s|-1) = D_{n-1}(s|-1) - (-1)^{n+s} D_{n-2}(s|-1)$$

$$\Delta_n(s|-1) = \Delta_{n-1}(s|-1) + (-1)^{n+s} \Delta_{n-2}(s|-1).$$

Replacing n by $n-1$ and $n-2$ and eliminating the terms with subscripts $n-1$ and $n-3$ we get

$$D_n(s|-1) = D_{n-2}(s|-1) + D_{n-4}(s|-1)$$

$$\Delta_n(s|-1) = \Delta_{n-2}(s|-1) + \Delta_{n-4}(s|-1)$$

both being recurrences of Fibonacci type. From this it follows from $D_2(s|-1) = 1 - (-1)^s$, $\Delta_2(s|-1) = 1 + (-1)^s$ that

$$D_{2m}(s|-1) = F_{m+1} - (-1)^s F_m$$

$$D_{2m+1}(s|-1) = F_{m+1}$$

$$\Delta_{2m}(s|-1) = F_{m+1} + (-1)^s F_m.$$

$$\Delta_{2m+1}(s|-1) = F_{m+1}$$

so the theorem follows for $k = 2$.

For $k \geqslant 3$ the proof involves "multisectioning." In (13) we set $x = \theta$ and multiply both sides by

$$(1-\theta z)(1-\theta^2 z)\cdots(1-\theta^{k-1}z) = (1-z^k)/(1-z)$$

obtaining

$$(1-z^k)G(z) = \sum_{m=0}^{k-1} q^m \theta^{m(m+s-1)} z^{2m} \prod_{\nu=m+1}^{k-1} (1-\theta^\nu z) + q^k z^{2k} G(z).$$

That is

(14) $$(1-z^k-q^k z^{2k})G(z) = P(z)$$

where $P(z)$ is a polynomial of degree $2k - 2$ at most.

Now let j be any non-negative integer less than k. If in (14) we replace z by $\theta^i z$ and multiply by θ^{-ij} and finally sum over $i = 0(1)k - 1$ we obtain

(15) $$(1-z^k-q^k z^{2k}) \sum_{i=0}^{k-1} \theta^{-ij} G(\theta^i z) = \sum_{i=0}^{k-1} \theta^{-ij} P(\theta^i z).$$

Next, let

$$Y_m = Y_m(j) = w_{j+mk}.$$

Then the sum on the left of (15) becomes

$$\sum_{n=0}^{\infty} w_n z^n \sum_{i=0}^{k-1} \theta^{i(n-j)} = k \sum_{n \equiv j \pmod k} w_n z^n = k \sum_{m=0}^{\infty} Y_m(j) z^{j+mk}.$$

The sum on the right of (15) is therefore z^j times a polynomial in z^k and so looks like $kz^j(A+Bz^k)$. Substituting into (15) and cancelling kz^j we have

(16) $$(1-z^k-q^k z^{2k}) \sum_{m=0}^{\infty} Y_m z^{mk} = A + Bz^k.$$

The left side can be written

$$Y_0 + (Y_1-Y_0)z^k + \sum_{m=2}^{\infty} (Y_m-Y_{m-1}-q^k Y_{m-2}) z^{mk}.$$

Hence

$$Y_m = Y_{m-1} + q^k Y_{m-2} \qquad (m \geqslant 2).$$

That is

(17) $$w_{j+mk} = w_{j+(m-1)k} + q^k w_{j+(m-2)k} \qquad (j=0(1)k-1).$$

If we set $q = 1$ so that by (12)

$$w_n = \Delta_n(s|\theta)$$

then (17) becomes

$$\Delta_{j+mk}(s|\theta) = \Delta_{j+(m-1)k}(s|\theta) + \Delta_{j+(m-2)k}(s|\theta).$$

Hence

$$\Delta_{j+mk}(s|\theta) = c_1\alpha^m + c_2\beta^m \sim c_1\alpha^m.$$

That is $|\Delta_n(s|\theta)|$ tends to infinity like $\alpha^{n/k}$. If we set $q = -1$ so th

$$w_n = D_n(s|\theta)$$

and if k is even we obtain the same asymptotic behavior for D_n. However if k is odd we obtain

$$D_{j+mk}(s|\theta) = D_{j+(m-1)k}(s|\theta) - D_{j+(m-2)k}(s|\theta).$$

Hence

$$D_{j+mk} = C_1\varepsilon^m + C_2\varepsilon^m$$

where $\varepsilon = e^{2\pi i/6}$. Thus D_{j+mk} is a periodic function of m of period 6. Finally $D_n(s|\theta)$ is a periodic function of n of period $6k$.

This completes the proof of the theorem.

Equation (16) is the source of much more explicit information th that given in Theorem 3. All one needs are the initial values w_j and w_{j+k} to obtain an explicit formula for w_{j+mk} as a function of m and in fact as a linear combination of the F_n or F_n' as we have already done for $k = 1$ and $k = 2$. The following are a few examples.

$$k = 3, \qquad \theta = e^{2\pi i/3} = \omega.$$

$$D_{3m}(s|\omega) = F_{m+1}' + \omega^s F_m' \qquad \Delta_{3m}(s|\omega) = F_{m+1} - \omega^s F_m$$

$$D_{3m+1}(s|\omega) = F_{m+1}' + \omega^{2s-1}F_m' \qquad \Delta_{3m+1}(s|\omega) = F_{m+1} + \omega^{2s-1}F_m$$

$$D_{3m+2}(s|\omega) = (1-\omega^s)F_{m+1}' \qquad \Delta_{3m+2}(s|\omega) = (1+\omega^s)F_{m+1}$$

For $k = 4$, $\theta = i$

$$D_{4m}(s|i) = F_{m+1} - (i^{s+1}+i^{2s})F_m \qquad \Delta_{4m}(s|i) = F_{m+1} + (i^{s+1}-i^{2s})F_m$$

$$D_{4m+1}(s|i) = F_{m+1} - i^{2s+1}F_m \qquad \Delta_{4m+1}(s|i) = F_{m+1} - i^{2s+1}F_m$$

$$D_{4m+2}(s|i) = (1-i^s)F_{m+1} + i^{3s}F_m \qquad \Delta_{4m+2}(s|i) = (1+i^s)F_{m+1} - i^{3s}F_m$$

$$D_{4m+3}(s|i) = (1-i^s-i^{s+1})F_{m+1} \qquad \Delta_{4m+3}(s|i) = (1+i^s+i^{s+1})F_{m+1}$$

Here are some sample formulas for $s = 1$ in which θ is a primitive kth root of unity

$k = 5$: $\quad D_{5m+4}(1|\theta) = -2(1+\theta^4)F_{m+1}' \qquad \Delta_{5m+4}(1|\theta) = 0$

$k = 6$: $\quad D_{6m+3}(1|\theta) = -2\theta^2F_{m+1} + F_m \qquad \Delta_{6m+3}(1|\theta) = 2\theta F_{m+1} + F_m$

$k = 8$: $\quad D_{8m+7}(1|\theta) = (4-\theta-2\theta^2-4\theta^3)F_{m+1} \qquad \Delta_{8m+7}(1|\theta) = \theta F_{m+1}$

$k = 14$: $\quad \Delta_{14m+6}(1|\theta) = 2\theta^5F_{m+3} - 2(1+\theta^2)F_{m+2} + (2\theta+3\theta^3)F_{m+1} + (\theta^3-2\theta^4)F_m$

$k = 15$: $\quad \Delta_{15m+13}(1|\theta) = (1+\theta^5-\theta^7)F_{m+1} + \theta F_m$

It will be observed in every case of

$$\Delta_{mk+k-1} \quad \text{or} \quad D_{mk+k-1}$$

that we have a polynomial in θ times F_{m+1} or F_{m+1}'. This is not just an accident but is a genuine theorem. This can be seen by taking $j = k - 1$ in the proof of Theorem 3. Since the degree of the right side of (15) must be $\leqslant 2k - 2$, the B must be zero in (16). This forces Y_1 and Y to be equal and leads at once to a solution of (17) in the form described above.

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AN OUTCROPPING OF COMBINATORICS IN NUMBER THEORY

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ABSTRACT. This paper exhibits an unexpected connection between problems in the theory of numbers having to do with factorization of integers in the rational and cyclotomic fields and combinatorial problems involving ordinary and supplementary difference sets.

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