Parallels Between Involutions and General Permutations

Aaron D. Jaggard

DIMACS
Rutgers University

Partially supported by NSF

Rutgers Experimental Mathematics Seminar
27 March 2008

Outline

Broad question

Question

In what ways do permutations in some class $\mathcal{P}_n \subseteq \mathcal{S}_n$ parallel permutations in some other class $\mathcal{Q}_n \subseteq \mathcal{S}_n$?

As a specific example:

Question

In what ways do involutions in $\mathcal{S}_n$ resemble permutations in general? (I.e., what does the imposition of symmetry do?)

Certainly not in all ways (*e.g.*, cycle-structure properties)

Here, we'll look at questions about 'permutation patterns' and involutions (permutations whose square is the identity).

Broad question

Question

In what ways do permutations in some class $\mathcal{P}_n \subseteq \mathcal{S}_n$ parallel permutations in some other class $\mathcal{Q}_n \subseteq \mathcal{S}_n$?

As a specific example:

Question

In what ways do involutions in $\mathcal{S}_n$ resemble permutations in general? (I.e., what does the imposition of symmetry do?)

Certainly not in all ways (*e.g.*, cycle-structure properties)
Here, we'll look at questions about 'permutation patterns' and involutions (permutations whose square is the identity).

Permutation patterns and pattern avoidance

The *pattern* of 7351 is 4231.

Definition

In general, the pattern of a word *w* of *j* distinct letters is the order-preserving relabeling of *w* with $\{1, \ldots, j\}$.

Definition

$\pi = \pi_1 \ldots \pi_n \in \mathcal{S}_n$ *contains* the pattern $\tau \in \mathcal{S}_k$ if there is a subsequence $\pi_{i_1} \ldots \pi_{i_k}$ of π whose pattern equals τ. Otherwise, π *avoids* τ.

4736521 contains the pattern 4231.

Permutation patterns and pattern avoidance

The *pattern* of 7351 is 4231.

Definition

In general, the pattern of a word *w* of *j* distinct letters is the order-preserving relabeling of *w* with $\{1, \ldots, j\}$.

Definition

$\pi = \pi_1 \ldots \pi_n \in \mathcal{S}_n$ *contains* the pattern $\tau \in \mathcal{S}_k$ if there is a subsequence $\pi_{i_1} \ldots \pi_{i_k}$ of π whose pattern equals τ. Otherwise, π *avoids* τ.

4736521 contains the pattern 4231.

Permutation patterns and pattern avoidance

The *pattern* of 7351 is 4231.

Definition

In general, the pattern of a word *w* of *j* distinct letters is the order-preserving relabeling of *w* with $\{1, \ldots, j\}$.

Definition

$\pi = \pi_1 \ldots \pi_n \in \mathcal{S}_n$ *contains* the pattern $\tau \in \mathcal{S}_k$ if there is a subsequence $\pi_{i_1} \ldots \pi_{i_k}$ of π whose pattern equals τ. Otherwise, π *avoids* τ.

4736521 contains the pattern 4231.

Permutation patterns and pattern avoidance

The *pattern* of 7351 is 4231.

Definition

In general, the pattern of a word w of j distinct letters is the order-preserving relabeling of w with $\{1, \ldots, j\}$.

Definition

$\pi = \pi_1 \ldots \pi_n \in \mathcal{S}_n$ *contains* the pattern $\tau \in \mathcal{S}_k$ if there is a subsequence $\pi_{i_1} \ldots \pi_{i_k}$ of π whose pattern equals τ. Otherwise, π *avoids* τ.

4$\underline{73}$6$\underline{5}$2$\underline{1}$ contains the pattern 4231.

$\mathcal{P}_n$-Wilf-equivalence

Definition

For $\mathcal{P}_n \subseteq \mathcal{S}_n$, let $\mathcal{P}_n(\alpha)$ be the number of permutations in $\mathcal{P}_n$ that avoid the pattern α. Let $\alpha \sim_{\mathcal{P}} \beta$ if $\mathcal{P}_n(\alpha) = \mathcal{P}_n(\beta)$ for every n. In this case we say that α and β are $\mathcal{P}_n$-*Wilf-equivalent* (or just *Wilf-equivalent* if $\mathcal{P}_n = \mathcal{S}_n$).

This naturally leads to two types of questions.

Two types of questions

Enumerative:

Question

For a family of permutations $\{\mathcal{P}_n\}_n$ ($\mathcal{P}_n \subseteq \mathcal{S}_n$) and a pattern α, what is the sequence $\{\mathcal{P}_n(\alpha)\}_n$?

Algebraic:

Question

What are the $\sim_{\mathcal{P}}$-classes of $\mathcal{S}_k$? For two different families $\{\mathcal{P}_n\}_n$ and $\{\mathcal{Q}_n\}_n$, how do the $\sim_{\mathcal{P}}$-classes of $\mathcal{S}_k$ compare to the $\sim_{\mathcal{Q}}$-classes of $\mathcal{S}_k$?

We'll focus on the algebraic questions.

Two types of questions

Enumerative:

Question

For a family of permutations $\{\mathcal{P}_n\}_n$ ($\mathcal{P}_n \subseteq \mathcal{S}_n$) and a pattern α, what is the sequence $\{\mathcal{P}_n(\alpha)\}_n$?

Algebraic:

Question

What are the $\sim_{\mathcal{P}}$-classes of $\mathcal{S}_k$? For two different families $\{\mathcal{P}_n\}_n$ and $\{\mathcal{Q}_n\}_n$, how do the $\sim_{\mathcal{P}}$-classes of $\mathcal{S}_k$ compare to the $\sim_{\mathcal{Q}}$-classes of $\mathcal{S}_k$?

We'll focus on the algebraic questions.

Two types of questions

Enumerative:

Question

For a family of permutations $\{\mathcal{P}_n\}_n$ ($\mathcal{P}_n \subseteq \mathcal{S}_n$) and a pattern α, what is the sequence $\{\mathcal{P}_n(\alpha)\}_n$?

Algebraic:

Question

What are the $\sim_{\mathcal{P}}$-classes of $\mathcal{S}_k$? For two different families $\{\mathcal{P}_n\}_n$ and $\{\mathcal{Q}_n\}_n$, how do the $\sim_{\mathcal{P}}$-classes of $\mathcal{S}_k$ compare to the $\sim_{\mathcal{Q}}$-classes of $\mathcal{S}_k$?

We'll focus on the algebraic questions.

Comparison to Wilf-equivalence

As with Wilf-equivalence, some $\mathcal{I}_n$-Wilf-equivalences (or 'involution-Wilf-equivalences') follow trivially from symmetry. However, the allowed symmetry operations are reduced because they must respect the symmetry of involutions.

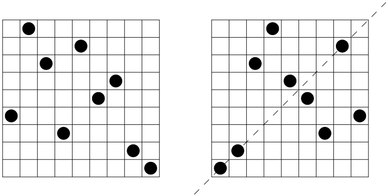

Figure: A permutation is an involution iff it is symmetric.

Initial results for involutions I

Theorem (Simion and Schmidt, 1985)

For $\tau \in \{123, 132, 213, 321\}$,

$$\mathcal{I}_n(\tau) = \binom{n}{\lfloor n/2 \rfloor}$$

and for $\tau \in \{231, 312\}$,

$$\mathcal{I}_n(\tau) = 2^{n-1}.$$

Note the contrast to single $\sim_{\mathcal{S}}$-class in $\mathcal{S}_3$.

Initial results for involutions II

Theorem (Regev, 1981)

$$\mathcal{I}_n(1234) = M_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} \binom{2i}{i} \frac{1}{i+1}$$

- Regev also gave asymptotics for $\mathcal{I}_n(12 \dots k)$ as $n \to \infty$.
- Gessel has given a determinantal formula for $\mathcal{I}_n(12 \dots k)$.
- $\mathcal{I}_n(12 \dots k)$ of interest because of Young tableaux.

Initial results for involutions II

Theorem (Regev, 1981)

$$\mathcal{I}_n(1234) = M_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} \binom{2i}{i} \frac{1}{i+1}$$

- Regev also gave asymptotics for $\mathcal{I}_n(12\dots k)$ as $n \to \infty$.
- Gessel has given a determinantal formula for $\mathcal{I}_n(12\dots k)$.
- $\mathcal{I}_n(12\dots k)$ of interest because of Young tableaux.

Initial results for involutions III

Theorem (Gouyou-Beauchamps, 1989)

$$\mathcal{I}_n(12345) = \begin{cases} C_k C_k, & n = 2k-1 \\ C_k C_{k+1}, & n = 2k \end{cases}$$

where C_k is the k^{th} Catalan number.

Theorem (Gouyou-Beauchamps, 1989)

$$\mathcal{I}_n(123456) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{3!n!(2i+2)!}{(n-2i)!i!(i+1)!(i+2)!(i+3)!}$$

Early algebraic results

Theorem (Guibert, 1995)

$$3412 \sim_{\mathcal{I}} 4321 \textit{ and } 2143 \sim_{\mathcal{I}} 1243$$

Later, one conjecture of Guibert was answered using generating trees.

Theorem (Guibert, Pergola, Pinzani 2001)

$$1234 \sim_{\mathcal{I}} 2143$$

This left one conjecture of Guibert open: whether or not $1432 \sim_{\mathcal{I}} 1234$.

Early algebraic results

Theorem (Guibert, 1995)

$$3412 \sim_{\mathcal{I}} 4321 \text{ and } 2143 \sim_{\mathcal{I}} 1243$$

Later, one conjecture of Guibert was answered using generating trees.

Theorem (Guibert, Pergola, Pinzani 2001)

$$1234 \sim_{\mathcal{I}} 2143$$

This left one conjecture of Guibert open: whether or not $1432 \sim_{\mathcal{I}} 1234$.

Early algebraic results

Theorem (Guibert, 1995)

$3412 \sim_{\mathcal{I}} 4321$ *and* $2143 \sim_{\mathcal{I}} 1243$

Later, one conjecture of Guibert was answered using generating trees.

Theorem (Guibert, Pergola, Pinzani 2001)

$1234 \sim_{\mathcal{I}} 2143$

This left one conjecture of Guibert open: whether or not $1432 \sim_{\mathcal{I}} 1234$.

Prefix-exchanging results

Theorem (J.)

For every permutation $\tau_3, \ldots, \tau_n$ *of* $\{3, \ldots, n\}$,

$$12\tau_3 \ldots \tau_n \sim_{\mathcal{I}} 21\tau_3 \ldots \tau_n$$

For every permutation $\tau_4, \ldots, \tau_n$ *of* $\{4, \ldots, n\}$

$$123\tau_4 \ldots \tau_n \sim_{\mathcal{I}} 321\tau_4 \ldots \tau_n$$

The analogous results for $\sim_{\mathcal{S}}$-equivalence were due to West (1990) and Babson and West (2000).

Conjectured that the prefixes $12 \ldots k$ and $k \ldots 21$ may be exchanged as well; the analogous result for $\sim_{\mathcal{S}}$-equivalence is due to Backelin, West, and Xin (2007).

Prefix-exchanging results

Theorem (J.)

For every permutation $\tau_3, \ldots, \tau_n$ *of* $\{3, \ldots, n\}$,

$$12\tau_3 \ldots \tau_n \sim_{\mathcal{I}} 21\tau_3 \ldots \tau_n$$

For every permutation $\tau_4, \ldots, \tau_n$ *of* $\{4, \ldots, n\}$

$$123\tau_4 \ldots \tau_n \sim_{\mathcal{I}} 321\tau_4 \ldots \tau_n$$

The analogous results for $\sim_{\mathcal{S}}$-equivalence were due to West (1990) and Babson and West (2000).

Conjectured that the prefixes $12 \ldots k$ and $k \ldots 21$ may be exchanged as well; the analogous result for $\sim_{\mathcal{S}}$-equivalence is due to Backelin, West, and Xin (2007).

Prefix-exchanging results

Theorem (J.)

For every permutation $\tau_3, \ldots, \tau_n$ *of* $\{3, \ldots, n\}$,

$$12\tau_3 \ldots \tau_n \sim_{\mathcal{I}} 21\tau_3 \ldots \tau_n$$

For every permutation $\tau_4, \ldots, \tau_n$ *of* $\{4, \ldots, n\}$

$$123\tau_4 \ldots \tau_n \sim_{\mathcal{I}} 321\tau_4 \ldots \tau_n$$

The analogous results for $\sim_{\mathcal{S}}$-equivalence were due to West (1990) and Babson and West (2000).

Conjectured that the prefixes $12 \ldots k$ and $k \ldots 21$ may be exchanged as well; the analogous result for $\sim_{\mathcal{S}}$-equivalence is due to Backelin, West, and Xin (2007).

Generalizing this result

Theorem (Bousquet-Mélou & Steingrímsson)

For every permutation $\tau_{j+1}, \ldots, \tau_k$ of $\{j+1, \ldots, k\}$,
$12\ldots j\tau_{j+1}\ldots\tau_k \sim_{\mathcal{I}} j\ldots 21\tau_{j+1}\ldots\tau_k$.

Proved by showing that the iterated transformation used in [BWX] commutes with inverting a permutation, even though the transformation itself doesn't.

Generalizing this result

Theorem (Bousquet-Mélou & Steingrímsson)

For every permutation $\tau_{j+1}, \ldots, \tau_k$ *of* $\{j+1, \ldots, k\}$*,* $12\ldots j\tau_{j+1}\ldots\tau_k \sim_{\mathcal{I}} j\ldots 21\tau_{j+1}\ldots\tau_k$.

Proved by showing that the iterated transformation used in [BWX] commutes with inverting a permutation, even though the transformation itself doesn't.

Implications for $\sim_{\mathcal{I}}$-equivalence

Applying this to the symmetry class $\{1243, 2134\}$ we obtain the result of Guibert, Pergola, and Pinzani:

$$1234 \sim_{\mathcal{I}} 2143$$

We may also affirmatively answer Guibert's conjecture:

$$1234 \sim_{\mathcal{I}} 3214$$

This completes the classification of $\mathcal{S}_4$ according to $\sim_{\mathcal{I}}$-equivalence.

Implications for $\sim_{\mathcal{I}}$-equivalence

Applying this to the symmetry class $\{1243, 2134\}$ we obtain the result of Guibert, Pergola, and Pinzani:

$$1234 \sim_{\mathcal{I}} 2143$$

We may also affirmatively answer Guibert's conjecture:

$$1234 \sim_{\mathcal{I}} 3214$$

This completes the classification of $\mathcal{S}_4$ according to $\sim_{\mathcal{I}}$-equivalence.

Placements on shapes and patterns

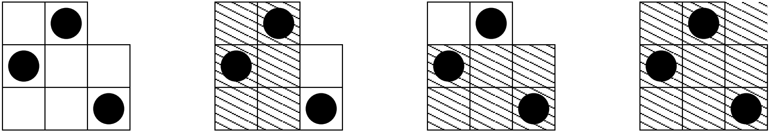

Figure: A placement on $(3, 3, 2)$ that contains 12 and 21 but not 231.

Self-conjugate shapes and symmetric placements

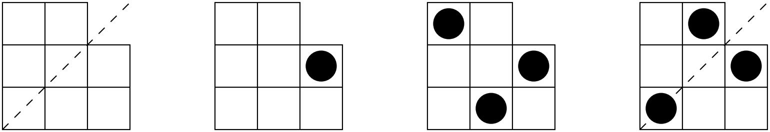

Figure: Four placements on the self-conjugate shape $(3, 3, 2)$.

From involutions to self-conjugate shapes with symmetric placements

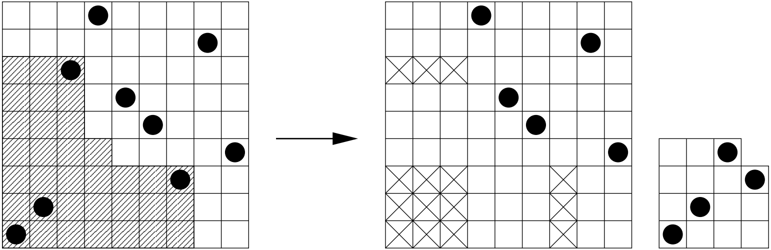

Figure: The involution shown contains 12354 iff the placement on $(4, 4, 4, 3)$ at the right contains 123.

We need the prefix to be an involution.

A useful theorem

Theorem (J.)

Let $\lambda_{sym}(T)$ be the number of symmetric full placements on the shape λ that avoid all of the patterns in the set T. Let α and β be involutions in $\mathcal{S}_j$. Let T_α be a set of patterns, each of which begins with the prefix α, and T_β similarly. If, for every self-conjugate shape λ, $\lambda_{sym}(\{\alpha\}) = \lambda_{sym}(\{\beta\})$, then for every self-conjugate shape μ,

$$\mu_{sym}(T_\alpha) = \mu_{sym}(T_\beta)$$

Exchanging 12 and 21

Backelin and West showed that there is a unique filling of any (fillable) shape that avoids 12, and a unique filling that avoids 21. These are necessarily symmetric if the shape is symmetric.

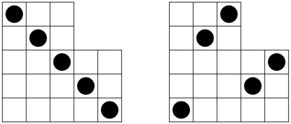

Figure: Starting from the top row, fill the box in either the leftmost (12-avoiding) or the rightmost (21-avoiding) column without a dot.

After these general results remaining question about $\sim_{\mathcal{I}}$-equivalences in $\mathcal{S}_5$ is:

Question

Does 54321 $\sim_{\mathcal{I}}$ 45312 *hold?*

Question

Does 654321 $\sim_{\mathcal{I}}$ 564312 *also hold (as suggested by numerical results)? If so, are these two cases of a more general result?*

These results are known for $\sim_{\mathcal{S}}$-equivalence, but do not follow from known $\sim_{\mathcal{I}}$ results.

The answer to all these questions: Yes!

Theorem

For every $k \geq 5$,

$$k(k-1)\ldots 321 \sim_{\mathcal{I}} (k-1)k(k-2)\ldots 312$$

In fact, this is is a corollary of a stronger theorem about *generating trees*.

The answer to all these questions: Yes!

Theorem

For every $k \geq 5$,

$$k(k-1)\ldots 321 \sim_{\mathcal{I}} (k-1)k(k-2)\ldots 312$$

In fact, this is is a corollary of a stronger theorem about *generating trees*.

Generating trees

Put a tree structure on the involutions avoiding a pattern τ

- If σ is an involution in $\mathcal{S}_n$ that avoids τ, then its parent π is the involution obtained by:
  1. Deleting the cycle containing n (either (n) or (jn))
  2. Taking the pattern of the resulting word
- The root of the tree is the empty permutation
- Find a way to label each node in the tree along with a rule that determines the labels of the children of a node with a given label

Theorem (J.–Marincel)

For every $k \geq 5$, the generating tree for involutions avoiding $k(k-1)\ldots 321$ is isomorphic to the generating tree for involutions avoiding $(k-1)k(k-2)\ldots 312$.

The number of involutions in $\mathcal{S}_n$ avoiding the pattern equals the number of nodes at depth n in the corresponding tree.

Corollary

$k(k-1)\ldots 321 \sim_{\mathcal{I}} (k-1)k(k-2)\ldots 312$.

Independently discovered (and generalized) by Dukes, Jelínek, Mansour, and Reifegerste.

Theorem (J.–Marincel)

For every $k \geq 5$, the generating tree for involutions avoiding $k(k-1)\dots321$ is isomorphic to the generating tree for involutions avoiding $(k-1)k(k-2)\dots312$.

The number of involutions in $\mathcal{S}_n$ avoiding the pattern equals the number of nodes at depth n in the corresponding tree.

Corollary

$k(k-1)\dots321 \sim_{\mathcal{I}} (k-1)k(k-2)\dots312$.

Independently discovered (and generalized) by Dukes, Jelínek, Mansour, and Reifegerste.

Defining labels

Given $\pi \in \mathcal{S}_n$, let p_i be the side of the largest square in the upper-right corner of the graph of π that does not contain a decreasing sequence of length $2i$ (k even, $1 \leq i \leq \frac{k}{2} - 1$) or $2i - 1$ (k odd, $1 \leq i \leq \frac{k-1}{2}$).

In the generating tree for involutions avoiding $k(k-1)\dots 321$, label π with $(n, p_1, p_2, \dots, p_{a-1}, p_m)$. [$m = \frac{k}{2} - 1$ or $m = \frac{k-1}{2}$]

In the generating tree for involutions avoiding $(k-1)k(k-2)\dots 312$, label $\pi \in \mathcal{S}_n$ with $(n, p_1, p_2, \dots, p_{m-1}, q_m)$, where $q_m + 1$ is the total number of depth-2 children of π.

The labels of the children of a node with label $(n, y_1, \ldots, y_m)$ are:

$$\{(n+1, w, y_2+1, \ldots, y_m+1)\} \cup \bigcup_{j=0}^{y_m} \{(n+2, z_1, \ldots, z_m)\},$$

where, in the label whose first component is $n+1$, w equals y_1+1 if k is even and 0 if k is odd, and in the label indexed by j:

$$z_i = \begin{cases} y_i + 2 & j \le y_{i-1} \\ j+1 & y_{i-1} < j \le y_i \;\; (j \le y_i \text{ for } i = 1) \\ y_i + 1 & y_i < j \end{cases}.$$

In the tree of involutions avoiding 654321, 53281764 has label $(n, p_1, p_2) = (8, 2, 4)$. Its depth-2 children and their labels are:

6329(10)18745	$(10, 3, 5)$
53291(10)8746	$(10, 3, 4)$
532918(10)647	$(10, 3, 6)$
5329176(10)48	$(10, 2, 6)$
53281764(10)9	$(10, 1, 6)$

In the tree of involutions avoiding 564312, 54821763 has label $(n, p_1, q_2) = (8, 2, 4)$. Its depth-2 children and their labels are:

(10)659328741	$(10, 3, 4)$
65(10)9218743	$(10, 3, 5)$
549218(10)637	$(10, 3, 6)$
5492176(10)38	$(10, 2, 6)$
54821763(10)9	$(10, 1, 6)$

Subsequence containment

Definition

$\pi = \pi_1 \ldots \pi_n \in \mathcal{S}_n$ *contains* the subsequence $\tau \in \mathcal{S}_k$ if there is a subsequence $\pi_{i_1} \ldots \pi_{i_k}$ of π such that $\pi_{i_j} = \tau_j$.

Unlike patterns, we care about the *exact* values!

Given $\tau \in \mathcal{S}_k$, it's trivial to see that the probability that $\pi \in \mathcal{S}_n$ (chosen u.a.r., $n \geq k$) contains τ as a subsequence is exactly $1/k!$

Subsequence containment

Definition

$\pi = \pi_1 \dots \pi_n \in \mathcal{S}_n$ *contains* the subsequence $\tau \in \mathcal{S}_k$ if there is a subsequence $\pi_{i_1} \dots \pi_{i_k}$ of π such that $\pi_{i_j} = \tau_j$.

Unlike patterns, we care about the *exact* values!

Given $\tau \in \mathcal{S}_k$, it's trivial to see that the probability that $\pi \in \mathcal{S}_n$ (chosen u.a.r., $n \geq k$) contains τ as a subsequence is exactly $1/k!$

Subsequence containment by involutions

Theorem (McKay, Morse, Wilf, 2002)

The probability that π (chosen u.a.r. from the involutions in $\mathcal{S}_n$, $n \geq k$) contains a subsequence $\tau \in \mathcal{S}_k$ equals $1/k! + o(1)$ as $n \to \infty$.

I.e., imposing symmetry doesn't really change the answer!

Counting the involutions containing a subsequence

Theorem (J., 2005)

For a fixed permutation $\tau = \tau_1\tau_2\ldots\tau_k \in \mathcal{S}_k$ and $n \geq k$, the number of involutions in $\mathcal{S}_n$ that contain τ as a subsequence equals

$$\sum{}' \binom{n-k}{k-j} t_{n-2k+j}$$

where the sum is taken over $j = 0$ and those $j \in [k]$ such that the pattern of $\tau_1 \ldots \tau_j$ is an involution in $\mathcal{S}_j$, and t_m equals the number of involutions in $\mathcal{S}_m$.

This allows us to sharpen the asymptotic results of [MMW]:

For $k > 2$, $\tau \in \mathcal{S}_k$, the probability as $n \to \infty$ that an involution $\pi \in \mathcal{S}_n$ contains τ as a subsequence is

$$\frac{1}{k!} - \frac{2}{3(k-3)!} n^{-3/2} + O(n^{-2})$$

if the pattern of $\tau_1\tau_2\tau_3$ is not an involution and

$$\frac{1}{k!} + \frac{1}{3(k-3)!} n^{-3/2} + O(n^{-2})$$

if it is.

Counting tableaux containing a subtableau

The RSK algorithm gives a bijection between standard Young tableaux of size n and the involutions in $\mathcal{S}_n$.

In a tableau corresponding to an involution $\pi \in \mathcal{S}_n$, the subtableau on $[k]$ depends only on the subsequence of π formed by the elements of $[k]$.

We may thus recover a formula of Sagan and Stanley counting the tableaux that contain a given subtableau.

Another notion of equivalence

Inspired by pattern avoidance, we make the following definition:

Definition

Two patterns α and β are *equivalent with respect to subsequence containment by involutions* iff, for every n, the number of involutions in $\mathcal{S}_n$ containing α as a subsequence equals the number containing β as a subsequence.

Characterizing this equivalence

Definition (j-set of a permutation)

Let $\mathcal{J}(\alpha) = \{j | \text{The pattern of } \alpha_1 \ldots \alpha_j \text{ is an involution in } \mathcal{S}_j\}$

Because each term in the sum counting the involutions containing a particular subsequence is asymptotically smaller than the previous one, α and β are equivalent in this sense iff $\mathcal{J}(\alpha) = \mathcal{J}(\beta)$.

Preliminary results

Theorem (J.)

The number of $\tau \in \mathcal{S}_k$ for which $\mathcal{J}(\tau) = \{0, \ldots, k\}$ equals 2^{k-1}.

Also, if we assume $\{0, 1, 2, k\} \subseteq E \subseteq \{0, 1, \ldots, k\}$ and $|E| = k \geq 5$, then the number of $\tau \in \mathcal{S}_k$ for which $\mathcal{J}(\tau) = E$ equals 2^{k-3} if $k - 1 \notin E$ and 2^{k-4} if $k - 1 \in E$.

Question

What is the sequence $\{|\mathcal{J}(\mathcal{S}_k)|\}_{k\geq 3} = 2, 4, 8, 16, 30, 56, 102, \ldots$?

Preliminary results

Theorem (J.)

The number of $\tau \in \mathcal{S}_k$ for which $\mathcal{J}(\tau) = \{0, \ldots, k\}$ equals 2^{k-1}.

Also, if we assume $\{0, 1, 2, k\} \subseteq E \subseteq \{0, 1, \ldots, k\}$ and $|E| = k \geq 5$, then the number of $\tau \in \mathcal{S}_k$ for which $\mathcal{J}(\tau) = E$ equals 2^{k-3} if $k-1 \notin E$ and 2^{k-4} if $k-1 \in E$.

Question

What is the sequence $\{|\mathcal{J}(\mathcal{S}_k)|\}_{k\geq 3} = 2, 4, 8, 16, 30, 56, 102, \ldots$?

Extensions

Theorem (Kim and Kim, 2007)

Assume that $\{j_1, j_2, \ldots, j_{r-1}\}$ *is a* j*-set and* $j_1 < \cdots < j_{r-1} < j_r$*. Then* $\{j_1, j_2, \ldots, j_{r-1}, j_r\}$ *is a* j*-set iff one of the following holds:*

1. $j_r - j_{r-1} = 1$
2. $j_{r-1} - j_{r-2} \neq 1$ *and* $j_r - j_{r-1} \geq j_{r-1} - j_{r-2}$
3. $j_{r-1} - j_{r-2} = 1$ *and* $j_r - j_{r-1} \geq j_{r-1} - j_{r-3}$

They also find a functional equation for the generating function of the number of j-sets in $\mathcal{S}_k$.

Conclusions

Parallel properties of involutions and general permutations

- Prefix-exchange results
- Other families of involution-Wilf-equivalences that correspond to Wilf-equivalences
- Subsequence containment—asymptotically the same