# Telescopers for 3D Walks via Residues

Shaoshi Chen

Department of Mathematics
North Carolina State University, Raleigh

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Joint with Manuel Kauers and Michael F. Singer

# Outline

- Motivation: enumerating 3D Walks.

- Integrability problems:

  Given $f \in K(y, z)$, decide whether

  $$f = D_y(g) + D_z(h) \quad \text{for some } g, h \in K(y, z).$$

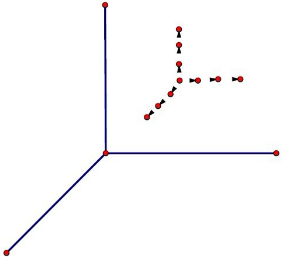
- Telescoping problems:

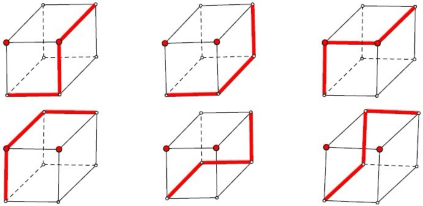
  Given $f \in k(x, y, z)$, find $L \in k(x)\langle D_x \rangle$ such that

  $$L(x, D_x)(f) = D_y(g) + D_z(h) \quad \text{for some } g, h \in k(x, y, z).$$

# Enumerating 3D Rook Walks

The Rook moves in a straight line as below in first quadrant of the 3D space.





R(1) = 6

$R(n)$: The number of different Rook walks from $(0,0,0)$ to $(n,n,n)$.

# 3D-diagonals

$f(m, n, k)$: the number of different Rook walks from $(0, 0, 0)$ to $(m, n, k)$.

$$F(x, y, z) = \sum_{m,n\geq 0} f(m, n, k)\, x^m y^n z^k = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y} - \frac{z}{1-z}}.$$

The diagonal of $F(x, y, z)$ is

$$\operatorname{diag}(F) := \sum_{n\geq 0} \underbrace{f(n, n, n)}_{R(n)}\, x^n.$$

Lemma: Let $\tilde{F} := (yz)^{-1} \cdot F(y, z/y, x/z)$ and $L(x, D_x) \in \mathbb{F}(x)\langle D_x\rangle$. Then

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(\tilde{F}) = D_y(G) + D_z(H) \quad \text{with } G, H \in \mathbb{F}(x, y, z) \;\Rightarrow\; L(\operatorname{diag}(F)) = 0.$$

# Telescoping Problems

Telescopers for trivariate rational functions:

Given $f \in \mathbb{F}(x, y, z)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(f) = D_y(g) + D_z(h) \quad \text{for some } g, h \in \mathbb{F}(x, y, z).$$

Telescopers for bivariate algebraic functions:

Given $\alpha(x, y)$ algebraic over $\mathbb{F}(x, y)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(\alpha) = D_y(\beta) \quad \text{for some algebraic } \beta(x, y) \text{ over } \mathbb{F}(x, y).$$

Goal: The two telescoping problems above are equivalent!

# Integrability Problems

Rational Integrability:

Given $f(y, z) \in \mathbb{E}(y, z)$, decide

$$f = D_y(g) + D_z(h) \quad \text{for some } g, h \in \mathbb{E}(y, z).$$

If such $g, h$ exist, we say that $f$ is rational Integrable w.r.t. $y$ and $z$.

Algebraic Integrability:

Given $\alpha(y)$ algebraic over $\mathbb{E}(y)$, decide

$$\alpha = D_y(\beta) \quad \text{for some algebraic } \beta \text{ over } \mathbb{E}(y).$$

If such $\beta$ exists, we say that $\alpha$ is algebraic Integrable w.r.t. $y$.

Goal: The two Integrable problems above are equivalent!

# Residues

Definition. Let $f \in \mathbb{F}(x, y)(z)$. The residue of $f$ at $\beta_i$ w.r.t. $z$, denoted by $\operatorname{res}_z(f, \beta_i)$, is the coefficient $\alpha_{i,1}$ in

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(z - \beta_i)^j}, \quad \text{where } p \in \mathbb{F}(x, y)[z],\ \alpha_{i,j}, \beta_i \in \overline{\mathbb{F}(x, y)}.$$

Lemma. Let $f \in \mathbb{F}(x, y)(z)$ and $\beta \in \overline{\mathbb{F}(x, y)}$.

- $\partial(\operatorname{res}_z(f, \beta)) = \operatorname{res}_z(\partial(f), \beta)$ with $\partial \in \{D_x, D_y\}$.
- $f = D_z(g) \quad \Leftrightarrow \quad$ All residues of $f$ w.r.t. $z$ are zero.

Remark. The second assertion is not true for algebraic functions!!!

# Equivalence between Two Integrability Problems

Theorem (PicardSimart1897). Let $f = A/B \in \mathbb{F}(x)(y,z)$. Then

$$f = D_y(g) + D_z(h) \;\Leftrightarrow\; \mathrm{res}_z(f,\beta) = D_y(\gamma_\beta) \text{ for all } \beta \text{ s.t. } B(\beta) = 0.$$

Example 1. Let $f = (x+y+z)^{-1}$. Since $\mathrm{res}_z(f, -x-y) = 1 = D_y(y)$, $f$ is rational Integrable w.r.t. $y$ and $z$. In fact,

$$f = D_y\left(\frac{x+y}{x+y+z}\right) + D_z\left(-\frac{x+y}{x+y+z}\right).$$

Example 2. Let $f = (xyz)^{-1}$. Since $\mathrm{res}_z(f,0) = (xy)^{-1}$ is not algebraic integrable, $f$ is not rational Integrable w.r.t. $y$ and $z$.

# Equivalence between Two Telescoping Problems

Theorem (Telescoping). Let $f \in \mathbb{F}(x, y, z)$ and $L \in \mathbb{F}(x)\langle D_x \rangle$. Then

$$L(x, D_x) \text{ is a telescoper for } f \text{ w.r.t. } y \text{ and } z$$

$$\Updownarrow$$

$$L(x, D_x) \text{ is a telescoper for every residue of } f \text{ w.r.t. } z$$

Remark.

$$L_i(x, D_x)(\alpha_i) = D_y(\beta_i), \; 1 \leq i \leq n$$

$$\Downarrow$$

$$L = \text{LCLM}(L_1, L_2, \ldots, L_n) \text{ is a telescoper for all } \alpha_i.$$

# Differentials and Residues

Let $K = \mathbb{F}(x, y)(\alpha)$ where $\alpha$ is an algebraic function over $\mathbb{F}(x, y)$. Think of $\alpha(x, y)$ as a parameterized family of algebraic functions of $y$ (with parameter $x$).

Differentials.

$$\Omega_{K/\mathbb{F}(x)} := \{\beta\, dy \mid \beta \in K\}.$$

- $df = 0$ for all $f \in \mathbb{F}(x)$ and $D_x(\beta dy) = D_x(\beta) dy$.

Residues. Let $\mathcal{P}$ be a place of $K$ (with no ramification). Then any $\beta \in K$ has a $\mathcal{P}$-adic expansion

$$\beta = \sum_{i \geq \rho} a_i t^i, \quad \text{where } \rho \in \mathbb{Z},\ a_i \in \overline{\mathbb{F}(x)} \text{ and } t \in K.$$

The residues of $\beta$ at $\mathcal{P}$ is $a_{-1}$, denoted by $\operatorname{res}_{\mathcal{P}}(\beta)$.

- $\operatorname{res}_{\mathcal{P}}(D_x(\beta)) = D_x(\operatorname{res}_{\mathcal{P}}(\beta))$.

# Differential Equations for Residues

Let $K = \mathbb{F}(x, y)(\alpha)$ and $\beta = A/B$ with $A \in \mathbb{F}(x)[y, \alpha]$ and $B \in \mathbb{F}(x)[y]$.
Let $B^*$ be the squarefree part of $B$ w.r.t. $y$.

Theorem. There exists $L \in \mathbb{F}(x)\langle D_x \rangle$ such that all residues of $L(\alpha)$ are zero and

$$\deg_{D_x}(L) \leq [K : \mathbb{F}(x, y)] \cdot \deg_y(B^*).$$

Definition. A differential $\omega \in \Omega_{K/\mathbb{F}(x)}$ is of second kind if all residues of $\omega$ are zero.

Lemma.

- If $\omega$ is exact i.e. $\omega = d(\beta)$, then $\omega$ is of second kind.
- Let $\Phi_{K/\mathbb{F}(x)} := \{\text{differentials of second kind}\}/\{\text{exact differentials}\}$. Then

$$\dim_{\mathbb{F}(x)}(\Phi_{K/\mathbb{F}(x)}) = 2 \cdot \text{genus}(\mathrm{K}).$$

# Telescopers for Bivariate Algebraic Functions

Algorithm. Given $\alpha(x,y)$ algebraic over $\mathbb{F}(x,y)$, do

1. Compute $L_1 \in \mathbb{F}(x)\langle D_x \rangle$ such that $\omega = L_1(\alpha)\,dy$ is of second kind.
2. Find $a_0, \ldots, a_{2g} \in \mathbb{F}(x)$ with $g := \text{genus}(K)$ with $K = \mathbb{F}(x,y)(\alpha)$, not all zero, such that

$$a_{2g} D_x^{2g}(\omega) + \cdots + a_0 \omega = d(\beta) \quad \text{for some } \beta \in K.$$

Remark.

- If $\alpha \in \mathbb{F}(x,y)$, Step 2 is not needed since $g = 0$.
- If $\omega$ is of second kind, so is $D_x^i(\omega)$ for all $i \in \mathbb{N}$.

# Telescopers for 3D Rook Walks

Transformation. $F = P/Q := (yz)^{-1} f(y, z/y, x/z)$.

$$\frac{P}{Q} = \frac{(-1+y)(y-z)(-z+x)}{zy\left(zy - 2\,yx - 2\,z^2 + 3\,xz - 2\,y^2z + 3\,y^2x + 3\,z^2y - 4\,zyx\right)}$$

Residues. Roots of $R(x, y, u) := \mathrm{Resultant}_z(Q, P - u \cdot D_z(Q))$ are

$$r_1 = \frac{y-1}{y(3y-2)},\ \ r_2 = -r_3 = \frac{(y-1)^2}{y(3y-2)\sqrt{-4y^3 + 16xy^2 + 4y^2 - y - 24xy + 9x}}.$$

Telescopers. $L_1 = D_x$ and $L_2 = L_3$ with

$$L_2 = {D_x}^3 + \frac{\left(4608\,x^4 - 6372\,x^3 + 813\,x^2 + 514\,x - 4\right) {D_x}^2}{x\left(-2 + 121\,x + 475\,x^2 - 1746\,x^3 + 1152\,x^4\right)} + \frac{4\left(576\,x^3 - 801\,x^2 - 108\,x + 74\right) D_x}{x\left(-2 + 121\,x + 475\,x^2 - 1746\,x^3 + 1152\,x^4\right)}$$

# Recurrences for 3D Rook Walks

$L = \text{LCLM}(L_1, L_2, L_3)$ is a telescopers for $F(x, y, z)$.

$$\Downarrow$$

$$L(x, D_x)\left(\sum_n f(n,n,n)x^n\right) = 0$$

Recurrence. Let $r(n) := f(n,n,n)$. From $L(x, D_x)$ via gfun, we get

$(1152n^2 + 1152n^3)r(n) + (-7830n - 3204 - 6372n^2 - 1746n^3)r(n+1) + (2957n + 762 + 2238n^2 + 475n^3)r(n+2) + (4197n + 4698 + 1240n^2 + 121n^3)r(n+3) + (-22n^2 - 80n - 96 - 2n^3)r(n+4) = 0.$

With initial values $r(0) = 1, r(1) = 6, r(2) = 222, r(3) = 9918$, we get

$$1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, \ldots$$

# Implementation and Experiments

Timings. We compare different algorithms for examples in combinatorics.

| | Chyzak | Koutschan | Residue |
|---|---|---|---|
| 3D Rook 1 | 3.48 | 24.5 | 0.59 |
| 3D Rook 2 | 31 | 182 | 2.3 |
| 3D Queen 1 | 11805 | $>$ 30h | 1203 |
| 3D Queen 2 | 12109 | $>$ 30h | 1186 |
| Random example | 221 | 1232 | 26 |

Figure: Timings are in seconds.

For more examples, please visit

http://www.risc.jku.at/people/mkauers/residues/

# Discrete and $q$-discrete

Summability problem:

Given $f \in K(x, y)$, decide whether $\exists\ g, h \in K(x, y)$ s.t.

$$f = \underbrace{g(x+1, y) - g(x, y)}_{\Delta_x(g)} + \underbrace{h(x, y+1) - h(x, y)}_{\Delta_y(h)}.$$

$q$-Summability problem:

Given $f \in K(x, y)$, decide whether $\exists\ g, h \in K(x, y)$ s.t.

$$f = \underbrace{g(qx, y) - g(x, y)}_{\Delta_{q,x}(g)} + \underbrace{h(x, qy) - h(x, y)}_{\Delta_{q,y}(h)}.$$

# Reducing double sums into single ones

Tornheim's identity:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n)^t}=\zeta(t-1)-\zeta(t),\quad \text{where } t>2 \text{ and } \zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

This follows from the identity:

$$\frac{1}{(m+n/2)^3}=\Delta_n\left(\frac{n/2}{(m+n/2)^3}+\frac{(n+1)/2}{(m+(n+1)/2)^3}\right)+\Delta_m\left(\frac{-1-n/2}{(m+n/2)^3}\right)$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n/2)^3}=4\zeta(2)-\frac{9}{2}\zeta(3).$$

This follows from the identity:

$$\frac{1}{(m+n/2)^3}=\Delta_n\left(\frac{n/2}{(m+n/2)^3}+\frac{(n+1)/2}{(m+(n+1)/2)^3}\right)+\Delta_m\left(\frac{-1-n/2}{(m+n/2)^3}\right)$$

# Irrationality of double sums

Theorem. For any $q \in \mathbb{N}$ with $q > 1$, the number

$$\mu = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{q^n + q^m}$$

is irrational.

Proof. Note that

$$\frac{1}{q^n + q^m} = \Delta_n \left( \frac{q/(1-q)}{q^n + q^m} \right) + \Delta_m \left( \frac{q/(1-q)}{q^{n+1} + q^m} \right).$$

Then

$$\mu = \frac{1}{2(q-1)} + \frac{2}{q-1}\mu_0, \quad \text{where } \mu_0 = \sum_{n=1}^{\infty} \frac{1}{q^n + 1}.$$

Peter Borwein in 1992 showed that $\mu_0$ is irrational. So $\mu$ is irrational.