Problems in (CM) Celestial Mechanics

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(2017-1687=330 years). Given N mass points that obey Newton's equations.

Siegel 1955 & Siegel and Moser 1971 (Lectures CM): "Despite efforts by outstanding mathematicians for over 200 years, the problem for $N > 2$ remains unsolved to this day". "Complete behavior of solutions". Based on initial positions and velocities. Which solutions of Newton's equations are: bounded? unbounded? Periodic? Have Collisions? Among which Particles? Nature of singularities? I will discuss:

Total Collapse. All N particles Collide at one point. A second opposite scenario $\rightarrow$

Total Escape. An expanding universe. All N particles escape to infinity without collisions or other singularities.

Language: CM=N body problem=Newton's or Gravitational Equations.

Harry Pollard, CM, Carus Monographs, 1966, (corrected) 1976. A 3 chapters introduction to CM. Accessible text with exercises$\rightarrow$ The N=2 body problem is still open.

Newton's Equations of the N body problem

Notation: Transposition of a Matrix=$\dagger$, Euclidean Norm: $\|y\|^2 = y^\dagger y$, $y \in \mathbb{R}^m$, $\dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$.

The differential equations governing the position 3-D vectors $q_i := \begin{bmatrix} x_i \\ y_i \\ z_i` \end{bmatrix}$, of N point-masses m_i, $i = 1, \ldots, N$ moving in $\mathbb{R}^3$ under the influence of their mutual gravitation is

$$m_i \ddot{q}_i = \sum_{j \neq i} \frac{m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3},\ m_i > 0. \qquad (1)$$

(above:i fixed j varies) . What is $\frac{m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3}$? $\sum_{j \neq i} \frac{m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3}$? $m_i \ddot{q}_i$? Where is G?

Newton: "preposterous", Feynman's bewilderment.

Initial conditions that we may not know!

$$q_i(t_0) = \alpha_i,\ \dot{q}_i(t_0) = \beta_i,\ \alpha_i \neq \alpha_j,\ i \neq j,\ \alpha_i, \beta_i \in \mathbb{R}^3. \tag{2}$$

How many unknowns? $6N$. (1) Is an equation for $q = \left(q_1^\dagger, \ldots, q_N^\dagger\right)^\dagger \in \mathbb{R}^{3N}$ of second order. $\dot{q}_i(t) := \frac{dq_i}{dt}, \dot{q} = \left(\dot{q}_1^\dagger, \ldots, \dot{q}_N^\dagger\right)^\dagger \in \mathbb{R}^{3N}$ is also an unknown (1). If written as a first order system then $y := \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^{6N},\ \dot{y} = some\ f(y) \in \mathbb{R}^{6N}$.

Notation: $(t_{inf},\ t_{sup})$ =maximal interval of existence of a solution to an IVP of (1) . $\sigma = t_{sup}\, or\, t_{inf}$

Example: $1 \cdot \dot{y} = -\frac{1}{y}, y(0) = 1$. $y = 0$ is a singular point for the normalized differential equation. $y = \sqrt{1-2t}$ is the unique solution. $y\left(\frac{1}{2}\right) = 0$, and $\dot{y}\left(\frac{1}{2}\right) = -\infty$, so $(t_{inf} = -\infty, t_{sup} = \sigma = 1/2)$.

Theorem 1. *The IVP (1),(2) possesses a unique solution on some* $[a, b] \subset (t_{inf},\, t_{sup})$.

Theorem. *The system of equations has no critical points in* $\mathbb{R}^{6N}$*. Hint: Use homogeneity.* $\sum_{j \neq i} \frac{m_j(q_j - q_i)}{\|q_i - q_j\|^3} \neq 0.$

Singularities I. Collision and Non collision

Given

$$\dot{q}_i(t) := \frac{dq_i}{dt},\ \ddot{q}_i = \sum_{j\neq i} \frac{m_j(q_j - q_i)}{\|q_i - q_j\|^3},\ q = \left(q_1^\dagger, \ldots, q_N^\dagger\right)^\dagger \in \mathbb{R}^{3N}. \tag{3}$$

The equations of motion (1) are real analytic everywhere except where two or more of the particles occupy the same point in $\mathbb{R}^3$. For $i \neq j$, let

$$\Delta_{ij} = \{q | q_i = q_j\}, \quad \Delta = \bigcup_{1\leq i<j\leq N} \Delta_{ij}. \tag{4}$$

$\triangle$ is called the collision set.

Remark: Singularities of an ODE do not necessarily coincide with Singularities of the Solutions. Singularities of the NORMALIZED N body:

1) The "invisible" $t = \infty$, 2) The "Collision Set" Δ . These may or may not coincide with singularities of a particular solution.

Example: Singularities of the (normalized) ode $1 \cdot \dot{y} = \frac{y}{t}$, @ $t = 0$. However, the solutions to the IVP with $y(0) = 0$, are $y(t) = kt$, $k \in \mathbb{R}$, $(t_{inf} = -\infty,\ t_{sup} = \infty)$.

Definition. *Consider $(t_{inf},\ t_{sup} = \sigma)$. If $\sigma < \infty$, then $q(t)$ is said to experience a singularity at σ.*

Singularity of a solution at a finite point $t_{sup} = \sigma$ means: $lim_{t \to \sigma^-} q(t)$ or $lim_{t \to \sigma^-} \dot{q}(t)$ or both do not exist and may be unbounded.

Definition. *We say that σ is a Collision singularity of a solution of (3) if*

$$lim_{t \to \sigma^-} q_i(t) = lim_{t \to \sigma^-} q_j(t) \in \triangle_{ij},\ i \neq j.$$

Theorem 2. [Painlevé 1897] . *i) If $q(t)$ is singular at σ, then $q(t) \to \Delta$, as $t \to \sigma^-$.*

ii) For $N = 2, 3$, all singularities are Collision Singularities.

Definition. *If q experiences a singularity at σ, but $q(t)$ does not approach a specific point $\hat{q} \in \Delta$, then q has a non-collision singularity.*

Theorem 3. [von Zeipel, 1908],[H. *Sperling,1970], [McGehee ,1986] . A non-collision singularity can only occur if the system of particles becomes unbounded in finite time.*

Wintner (1941) and Pollard and Saari (1968) distrusted von Zeipel proof. McGehee (1986) showed that the initial argument had been correct.

[Poincare, Painleve]: Are there solutions with non-collision singularities? Solutions in $\mathbb{R}^d, d = 1, 2, 3$:

Theorem 4. *i) [Mather and McGehee 1975], Yes for $N = 4$, $d = 1$, COLLINEAR motion.*

ii) [J. Gerver,1991], Yes, for a $3N$ body motion in a plane with N very Large,$d = 2$.

iii) [Z. Xia, 1992] , Yes for $N = 5$, $d = 3$.

10 Integrals of motion

Example of an "integral" of a scalar differential equation

$$2\ddot{z} + 4z^3 = 0 \Rightarrow 2\ddot{z}\dot{z} + 4z^3\dot{z} = 0 \Longrightarrow \tag{5}$$

$$g(z(t), t) := \dot{z}^2(t) + z^4(t) = h.$$

$g(z, t)$ is an "integral" .

Let $q_k^\dagger =< x_k, y_k, z_k >$. 1 "integral" , for the total energy

$$T = \sum_{j=1}^{N} \frac{m_j}{2} \left\|\dot{q}_j(t)\right\|^2, U = \sum_{j<k} \frac{m_j m_k}{\|q_j - q_k\|} > 0,\ T - U = h. \tag{6}$$

T is the kinetic energy and $U = \sum_{j<k} \frac{m_j m_k}{\|q_j - q_k\|} > 0$ is the negative of the potential energy. The gradient notation for $U := U(q)$

$$\nabla_{q_k} U := < \frac{\partial U}{\partial x_k}, \frac{\partial U}{\partial y_k}, \frac{\partial U}{\partial z_k} >^{\dagger} = U_{q_k} = U_k = \nabla_k U$$

satisfies

$$m_k \ddot{q}_k = U_k = \sum_{j \neq k} \frac{m_k m_j (q_j - q_k)}{\|q_k - q_j\|^3} \Longrightarrow \tag{7}$$

$$\sum_{k=1}^{N} m_k \ddot{q}_k = \sum_{k=1}^{N} U_k = 0 \Longrightarrow \tag{8}$$

One vector equation of Conservation of Linear Momentum$\Longrightarrow$ 3 scalar equations of conservation of Linear Momentum

$$\sum_{k=1}^{N} m_k \dot{q}_k = a \Longrightarrow \tag{9}$$

$$\sum_{k=1}^{N} m_k q_k = at + b,\ a,\ b \in \mathbb{R}^3 \Rightarrow$$

6 scalar integrals of motion.

Proposition 1. *The center of mass q_c of the N bodies travels along a straight line.*

Proof. Indeed

$$q_c := \frac{\sum_{k=1}^{N} m_k q_k}{\sum_{k=1}^{N} m_k} = \frac{a}{\sum_{k=1}^{N} m_k} t + \frac{b}{\sum_{k=1}^{N} m_k}. \qquad (10)$$

□

Yet another vector equation $\Longrightarrow$ 3 scalar equations of the total angular momentum

$$\sum_{k=1}^{N} (q_k \times m_k \dot{q}_k) = c. \qquad (11)$$

All together 10 scalar algebraic equations in the $6N$ variables $\begin{bmatrix} q \\ \\ \dot{q} \end{bmatrix} \in \mathbb{R}^{6N}$.

What use? i) Simplification. Reduce the number of $6N$ scalar differential equations of N body problem by

10 to solve only $6N - 10$ scalar ODE's. In principle algebraic equations are simpler than ODE's. ii) Information on trajectories and SINGULARITIES for $N = 1, 2$.

Theorem 5. *Bruns, (1887-1888) :No additional algebraic integrals of CM equations exist that are independent of the 10 above.*

Definition 1. *A continuously differentiable (non constant) scalar function $g(q, \dot{q}, t) \in C^1(\mathbb{R}^{6N} \setminus \triangle, \mathbb{R})$ of the $6N + 1$ variables $(q, \dot{q}, t)$ is said to be an "integral" of (1) if $g(q(t), \dot{q}(t), t) = Constant$ for a solution $(q(t), \dot{q}(t))$ of (1).*

The power of T-U=h.

Singularities of the N- body Equations=Collision set $\Delta_{ij} = \{q | q_i - q_j = 0\}$, $\Delta = \bigcup_{1 \le i < j \le N} \Delta_{ij}$, coincide with singularities of solutions.

Theorem 6. **[Painlevé 1897]** . *i) If $q(t)$ is singular at σ, then $q(t) \to \Delta$, as $t \to \sigma^-$.*

Proof. A reoccuring argument

$$\sum_{j=1}^{N} \frac{m_j}{2} \|\dot{q}_j(t)\|^2 - \sum_{j<k} \frac{m_j m_k}{\|q_j(t) - q_k(t)\|} =$$

$$T(\dot{q}(t)) - U(q(t)) = h.$$

$T \ge 0, U > 0$, As $t \to \sigma^-$, $inf_{t,\, j \neq k} \|q_j(t) - q_k(t)\| = 0$ iff $sup \sum_{j=1}^{N} \frac{m_j}{2} \|\dot{q}_j(t)\|^2 = \infty$. $\square$

Oscar King of Sweden and Norway$\cap$Weierstrass$\cap$Mittag Leffler$\cap$Poincare$\cap$Sundman$\cap$Wang Qiu-Dong

Construct a series solution to the IVP of the N body problem VALID for ALL TIME.

See F. Diacu, *"The Solution of the n-body Problem,"* *Mathematical Intelligencer*, 18 (1996) 66-70.

Weierstrass$\Longrightarrow$ Mittag Leffler$\Longrightarrow$Oscar King of Sweden and Norway$\Longrightarrow$Prize$\Longrightarrow$Poincare (1889) for original and valuable ideas (Poincare did not solve the problem)

K. Sundman, 1913, 'Mémoire sur le problème des trois corps', Acta Math. 36, 105–179. N=3, Power series solution for

$$Angular\, Momentum = \sum_{j=1}^{3} q_j(t) \times m_j \dot{q}_j(t) \neq 0. \tag{12}$$

Wang Qiu-Dong, (1991). The global solution of the N-Body problem, Celestial Mechanics and Dynamical Astronomy. (Stays away from singularities).

Some tools and ingredients:

a) Construct a 1-1 continuously differentiable mapping from $t \in (t_{inf},\, t_{sup})$ onto $\tau \in (-\infty, \infty)$.

b) Consider the solutions as complex valued analytic solutions of the complex variable τ in a strip about the real τ axis in the complex plane and construct a conformal map of the strip onto the open Unit Disk$:= w \Bigg| |w| < 1$.

Summary of the arguments:

$q(t), t = t(\tau)$, $\tau = \tau(w) \Longrightarrow Q(w) := q(t(\tau(w)))$ is analytic in the unit disk and has a converging power series in w.

Advantages: Solution of an old famous problem.

Disadvantages: i) may have very slow convergence ii) Does not inform on the mechanism of collision or non collision singularities iii) Does not inform about asymptotics on semi infinite intervals of existence say $t \in (0, \infty)$.

Should we look for analogous results for any real analytic differential system?

Expanding Universe=Total Escape as $t \to \infty$.

Definition. The universe is expanding *if the N body problem possesses solutions $q_i(t), i = 1, \ldots, N$ that exist on a semi-infinite interval $[t_0, \infty)$, that satisfy $\lim_{t\to\infty} \|q_i(t) - q_j(t)\| = \infty$, $i \neq j$.*

Lemaître, **(1927),** appears to have been the first to notice that the Einstein field equations of General Relativity admit solutions that expand forever.

Hubble's observation, (1929): many galaxies are speeding away from us in the milky way, and from each other.$\Rightarrow$ we are living in an expanding universe.

Do Newton's gravitational equations support an expanding universe? All N bodies receding from each other (FREE of SINGULARITIES)?

Theorem 7. *[Bohlin, 1908]. The $N = 3$ body problem* *has for* t *large FORMAL solutions of the form*

$$q_i(t) = a_i t + b_i log\, t + c_i + P_i(\frac{1}{t}, \frac{log\, t}{t}), i = 1, 2, 3.$$

$P_i(z, w)$ *formal power series in two variables* z *and* w, $P_i(0, 0) = 0$.

Next

Theorem 8. *[Chazy, 1922]. Assume: CI)* $Energy = T - U = h > 0$, *CII) Solutions of the* $N = 3$ *body problem exists on some semi infinite interval FREE OF SINGULARITIES,*

Then, I) the N=3 body problem has solutions of the form

$$q_i = a_i t + b_i \log t + c_i + \delta_i(t),\ i = 1, 2, 3,\ lim_{t \to \infty} \delta_i(t) = 0. \tag{13}$$

II) $\|a_j - a_i\| \neq 0,\ i \neq j.$

III) $b_i = -\sum_{j \neq i} \frac{m_j (a_j - a_i)}{\|a_j - a_i\|^3},\ i = 1, 2, 3.$

Remarks: For t large. IF there are NO SINGULARITIES then the N bodies separate like II). Positions $q_i \sim a_i t$ are distinct and so are velocities $\dot{q}_i \sim a_i$.

According to Chazy, Poincare missed the term $b_i \log t$.

Theorem 9. *[Myself & Solomon, JDE, 2017]. Given any set of constant a_i , $c_i \in \mathbb{R}^3$, $i = 1, \ldots, N$, satisfying II). Then, the N body problem possesses unique vector solutions*

i) of the form

$$q_i = a_i t + b_i \log t + c_i + \delta_i(t) \; i = 1, \ldots, N, \; lim_{t \to \infty} \delta_i(t) = 0,$$

on a semi infinite interval $[t_5, \infty)$ where $t_5 > 0$ and $q_i \in C^\infty[t_5, \infty)$.

ii) (FREE OF SINGULARITIES)

iii) $Energy = T - U = h > 0$.

iv) The $3N$ coefficients b_i are uniquely determined by *the* $3N$ *coefficients* a_i *as follows*

$$b_i = -\sum_{j \neq i} \frac{m_j\,(a_j - a_i)}{\|a_j - a_i\|^3},\ i = 1,\,\ldots,\,N.$$

Proof. Integration of the N-body eq. by very fast converging series. □

Remark: e.g. Birkhoff, text 1966, Pollard,1967, Saari 1971, Marchal & Saari,1976, Mingarelli's 2005 assumed CII). Saari et al extended in various manners [Chazy, 1922] to the N body problem ASSUMING existence of solutions on a semi infinite interval FREE of SINGULARITIES rather than proving that.

Collisions and Total Collapse

Assume throughout given the IVP

$$\dot{q}_i(t) := \frac{dq_i}{dt},\ \ddot{q}_i = \sum_{j \neq i} \frac{m_j(q_j - q_i)}{\|q_i - q_j\|^3}, \tag{14}$$

$$q_i(t_0) = \alpha_i,\ \dot{q}_i(t_0) = \beta_i,\ \alpha_i \neq \alpha_j,\ i \neq j,\ \alpha_i, \beta_i \in \mathbb{R}^3. \tag{15}$$

Definition 2. *We say that a singularity of the solution of the IVP is due to a collision at time σ if there exist two indices $j \neq k$ such that $lim_{t \to \sigma^+} q_j(t) = lim_{t \to \sigma^+} q_k(t) = F \in \triangle$. (or σ^-).*

We say that total Collapse occurs if $lim_{t \to \sigma^-} q_k(t) = F \in \triangle, k = 1, 2, \cdots, N$.

Theorem. *Sundman (1906, 1907 & 1913).*

i) If total Collapse is to occur it will not take forever to happen.

ii) Total collapse cannot occur unless the angular momentum is zero.

Later we have

Theorem. *(Pollard & Saari (1968)). i) A singularity as $t \to \sigma = 0^+$ is due to a collision if and only if for some positive constant α*

$$U(t) = \sum_{j<k} \frac{m_j m_k}{\|q_j(t) - q_k(t)\|} \sim \alpha t^{-\frac{2}{3}},\, as\, t \to 0^+.$$

ii) A singularity as $t \to 0^+$ is due to a collision if and only if (the moment of inertia)

$$I(t) := \frac{1}{2} \sum_{k=1}^{N} m_k \left\| q_k(t) \right\|^2 = \mathcal{O}(1),\, as\, t \to 0^+.$$

Tools

Sundman, Chazy, Pollard, Pollard and Saari, Saari & Marchal,...

10 integrals of motion. Conservation of: momentum, angular momentum and Energy.

Fix origin of coordinate system at center of mass

$$q_c := \frac{\sum_{k=1}^{N} m_k q_k}{\sum_{k=1}^{N} m_k} \Rightarrow \hat{q}_k = q_k - q_c \Longleftrightarrow \{\hat{q}_k = 0 \Leftrightarrow q_k = q_c\}.$$

The $\frac{1}{2}$ moment of inertia $I(t) := \frac{1}{2}\sum_{k=1}^{N} m_k \left\|\hat{q}_k(t)\right\|^2$. After relabeling $I(t) := \frac{1}{2}\sum_{k=1}^{N} m_k \left\|q_k(t)\right\|^2$.

Total Collapse$\Longleftrightarrow$ all point masses converge on the center of mass

$$\Longleftrightarrow I(t) \to 0,\ as\, t \to \sigma^- \Longleftrightarrow q_k(t) \to 0,\ k = 1, 2, \cdots, N. \tag{16}$$

Lagrange -Jacobi Identity

$$\ddot{I} = 2T - U = T + h = U + 2h. \tag{17}$$

Sundman inequality with $c = \sum_{k=1}^{N}(q_k \times m_k \dot{q}_k)$

$$\|c\|^2 \leq 4I(\ddot{I} - h). \tag{18}$$

Tauberian Theorems.

McGehee (1974) Coordinates. An integration of the N-Body problem via new coordinates. $\Leftarrow$ Wang-(series solutions), Zia-(Existence of Non Collision singularities).

Simplest: The central force problem N=1.

Any hints for the very complicated N body problem?

$q := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, $\|q\| := \sqrt{q^t q} = \sqrt{x^2 + y^2 + z^2}$, $f(\|q\|) > 0$ is a scalar function

$$\ddot{q} = -f(\|q\|)[\|q\|^{-1} q], \; e.g. \, f(\|q\|) = \frac{\mu}{\|q\|^{\theta}}, \; \mu, \theta \in \mathbb{R}. \tag{19}$$

CM has a special central force

$$\ddot{q} = -\frac{\mu q}{\|q\|^3}, \; \mu > 0, \; f(\|q\|) = \frac{\mu}{\|q\|^2}. \tag{20}$$

Assume the center of the coordinates system is at $(0, 0, 0)$. How much do we know?

i) Conservation of Kinetic+Potential=h a constant.

$$\frac{\dot{q}^t \dot{q}}{2} - \frac{\mu}{\sqrt{q^t q}} = h. \qquad (21)$$

Exist two constant vectors of the motion, c and e that play a fundamental role in description of the trajectory of a particle in the central force problem.

ii) c the angular momentum vector that is conserved

$$q \times \dot{q} = c = constant\ vector$$

iii) If $c \neq 0$ then the motion of the particle with point mass m takes place in a plane passing through $(0,0,0)$ and $\perp$ to c.

Assume $c \neq 0$.

iv) If the position P of the particle on its trajectory is given in polar coordinates $(r = \|q\|, \theta)$ then upon proper choice of coordinates $q = (\|q\| cos(\theta), \|q\| sin(\theta), z =$

$0)$, $c = \|c\| (0, 0, 1)$, such that the motion takes place say in the xy plane and the vector c and the z coordinate has the same direction , then the rate at which the area is swept out by a radius vector emanating from $(0, 0, 0)$ is

Kepler's Second Law.

$$\frac{1}{2} \|q\| (\dot{\theta})^2 = \frac{1}{2} \|c\| . \tag{22}$$

v) The other constant vector of integration e (with e"eccentric axis") satisfies

$$\mu(e + \frac{q}{\|q\|}) = \dot{q} \times c, \; e \cdot q + \|q\| = \frac{\|c\|^2}{\mu}.$$

vi) The trajectory of a particle in a central field of gravitation is given in polar coordinates (with ω a constant) by

$$r = \|q\| = \frac{\frac{\|c\|^2}{\mu}}{1 + \|e\| \, cos(\theta - \omega)}.$$

vii) Kepler's First Law extended. The motion takes place in a plane and the trajectories are conic sections.

$\|e\| = 0 \Rightarrow r = \|q\| = \frac{\|c\|^2}{\mu} \Longrightarrow$ a circle.

$0 < \|e\| < 1 \Longrightarrow$ ellipse.

$\|e\| = 1 \Rightarrow$ a parabola

$\|e\| > 1 \Rightarrow$ a branch of hyperbola.

A focus of the conic coincides with $(0, 0, 0)$. Non of the cases in vii) coincides with a collision. Are collisions rare?

viii) Kepler's Third law. If a denotes the length of the semi major axis of the conic then

$$Area\ Ellipse = \pi a^2 (1 - \|e\|^2)^{\frac{1}{2}},\ Period = (\frac{2\pi}{\sqrt{\mu}})^{\frac{3}{2}}.$$

Remark: The impact of physical quantities like c=angular momentum and e =eccentric axis on the shape of a trajectory.

iv) If $c = 0$ then the motion of the particle takes place along a fixed straight line passing $(0, 0, 0)$. Degenerate motion. the only possibility for collision.

Most motion is singularities and/or collision free. What if $N > 1$?

Theorem 10. *Saari (1973) : Collision singularities in the N body problem are improbable. The set of initial conditions leading to collision has measure zero.*

Classification of Motion by h. Recall $\frac{\dot{q}^t \dot{q}}{2} - \frac{\mu}{\sqrt{q^t q}} = h.$

When combining the above (where $c \neq 0$) with

$$\mu^2(\|e\|^2 - 1) = 2h\,\|c\|^2 \Longrightarrow \tag{23}$$

$0 \leq \|e\| < 1$ Ellipse or circle. By (23) $\Longleftrightarrow h < 0$.

$\|e\| = 1 \Rightarrow$ a parabola. By (23) $\Longleftrightarrow h = 0$.

$\|e\| > 1 \Rightarrow$ a hyperbola. By (23) $\Longleftrightarrow h > 0$.

In the N body problem some researchers continue to classify motion by the sign of h.

Solutions in $\mathbb{R}^d$, $d = 1, 2, 3$.

A goal: "Explicit" e.g. $\dot{y} = y \Rightarrow y(t) = \mu e^t$. Reduction to algebraic equations

Euler (1767), $d = 1$, $N = 3$ bodies moving along a straight line.

Lagrange (1772), $d = 2$, $N = 3$ bodies $k = 1, 2, 3$ rotating in a plane with same constant angular velocity ω such that

$$x_k(t) = \xi_k cos\, \omega t - \eta_k sin\, \omega t,\ y_k(t) = \xi_k sin\, \omega t + \eta_k cos\, \omega t. \tag{24}$$

A configuration where the 3 bodies are positioned at the vertices of an equilateral triangle.

Central Configurations=CC

A central configuration is a special arrangement of the N point masses interacting by Newton's law of gravitation with the following property:

$$U_i = -\lambda m_i(q_i - s),\ \lambda \in \mathbb{R},\ s, q_i \in \mathbb{R}^d,\ d = 1, 2, 3, \cdots, \tag{25}$$

$$U_i = \sum_{j \neq i} \frac{m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3}.$$

An ALGEBRAIC EQUATION!

Proposition 2. *If (25) holds then s="weighted average of position vectors"=center of mass is*

$$s = q_c = \sum_{k=1}^{N} \left(\frac{m_k}{\sum_{k=1}^{N} m_k}\right) q_k,\ \lambda = \frac{U}{J},$$

where

$$U = \sum_{j<k} \frac{m_j m_k}{\|q_j - q_k\|} > 0,\ J = \sum_{k=1}^{N} m_k \left\|q_k(t) - q_c\right\|^2. \tag{26}$$

Open problem. Finite # Equivalence Classes of CC?

Call two configurations $q_i, \tilde{q}_i \in R^d, d = 1, 2, 3$ equivalent if there are constants $\alpha \in R, b \in R^d$ and an dxd orthogonal matrix Q such that

$$\tilde{q}_i = \alpha Q q_i + b, i = 1, \ldots, N. \tag{27}$$

.

Proposition 3. *If q satisfies*

$$\sum_{j \neq i} \frac{m_j (q_j - q_i)}{\|q_i - q_j\|^3} = -\lambda (q_i - s) \tag{28}$$

with constants λ, s then $\tilde{q}$ satisfies

$$\sum_{j \neq i} \frac{m_j (\tilde{q}_j - \tilde{q}_i)}{\|\tilde{q}_i - \tilde{q}_j\|^3} = -\tilde{\lambda} (\tilde{q}_i - \tilde{s}). \tag{29}$$

with constants $\tilde{\lambda} = |\alpha|^3 \lambda, \tilde{s} = s + b$ *. (28) and (29) have the same form. Therefore, we say that (28) is invariant under (27). Under Dilation* α*, Rotation* Q *and translation* b*.*

For the CC equations, this is sometimes called the Chazy-Wintner-Smale problem: given N positive masses, is the number of Equivalence classes of CC finite? [Smale singled out the planar case $d = 2$ as the 6-th of his problems for the twenty-first century.]

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