Proofs of Ramanujan series by the WZ-method

Jesús Guillera

Zaragoza (Spain)

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Introduction

In this talk we will use the Wilf-Zeilberger (WZ)-method to prove in an elementary way formulas like

$$\sum_{n=0}^{\infty} \left(\frac{-1}{16}\right)^n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} (51n+7) = \frac{12\sqrt{3}}{\pi},$$

or

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2+27n+3) = \frac{48}{\pi^2},$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$.

The first one is a Ramanujan-type series due to Chan, Liaw and Tan (2003), who proved it using elliptic modular functions.

All the known proofs of the second formula are based on WZ-pairs.

The Pochhammer symbol

The rising or sifting factorial (Pochhammer symbol) is defined by

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}, \qquad (0)_0 = 1.$$

If x is a positive integer, it reduces to

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1),$$

For $a = 1$, we have

$$(1)_n = n!,$$

and we see that the rising factorial generalize the ordinary factorial.

The series for $1/\pi$ of the form

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3} z^n (a+bn) = \frac{1}{\pi},$$

where $s = 1/2$, $1/4$, $1/3$, or $1/6$ and z, a, b are algebraic numbers, were discovered by S. Ramanujan, who gave 17 examples in 1914.

One of them is

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) = \frac{16}{\pi}.$$

It gives approximately $\log 64 \simeq 1.8$ digits of π per term.

Other series by Ramanujan

The most impressive series discovered by Ramanujan are:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{882^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (21460n + 1123) = \frac{3528}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{1}{99^{4n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (26390n + 1103) = \frac{9801\sqrt{2}}{4\pi},$$

which give almost 6 and 8 digits per term respectively.

J. and P. Borwein were the first to prove the 17 Ramanujan series by using the theory of elliptic modular functions and equations.

Rational and irrational Ramanujan series

Ramanujan only gives the following example of irrational series:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \frac{1}{2^{6n}} \left(\frac{\sqrt{5}-1}{2}\right)^{8n} \left[(42\sqrt{5}+30)n + (5\sqrt{5}-1)\right] = \frac{32}{\pi}.$$

The brothers D. and G. Chudnovsky proved the formula

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{53360^{3n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \frac{545140134n + 13591409}{426880} = \frac{\sqrt{10005}}{\pi},$$

which has the property of being the fastest possible rational series. This is so because for this series we have

$$b^2 = 163(1-z),$$

the greatest number for which $\mathbb{Q}(\sqrt{-r})$ has unique factorization.

Let $s_0 = 1/2$, $s_3 = 1 - s_1$, $s_4 = 1 - s_2$ and

$$\begin{aligned}(s_1, s_2) =& (1/2, 1/2), (1/2, 1/3), (1/2, 1/4), (1/2, 1/6), (1/3, 1/3), \\ & (1/3, 1/4), (1/3, 1/6), (1/4, 1/4), (1/4, 1/6), (1/6, 1/6), \\ & (1/5, 2/5), (1/8, 3/8), (1/10, 3/10), (1/12, 5/12).\end{aligned}$$

We will call Ramanujan-like series for $1/\pi^2$ to the series which are of the form

$$\sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i)_n}{(1)_n} \right] (a + bn + cn^2) = \frac{1}{\pi^2},$$

where z, a, b and c are algebraic numbers. Observe that now we have five rising factorials in the numerator instead of three.

The PSLQ algorithm

Let $(x_1, \dots x_n)$ be a vector of real numbers and write all the numbers the x_j with a precision of d decimal digits.

The PSLQ algorithm finds a vector $(a_1, \dots, a_n)$ of integers (with $a_j \neq 0$ for some j), such that:

$$a_1 x_1 + \cdots + a_n x_n = 0, \quad \text{(with a precision of d digits)},$$

and which has the smallest possible norm.

The PSLQ algorithm discovers identities but do not prove them.

Example: Let

$$f(j) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{kn}} n^j, \qquad k = 1, 2, 3, \dots$$

and look for integer relations among $f(0)$, $f(1)$, $f(2)$ and $1/\pi^2$.

The formulas we found and proved

With PSLQ we discovered the formulas

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{2n}} (20n^2 + 8n + 1) = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{1}{2^{4n}} (120n^2 + 34n + 3) = \frac{32}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (820n^2 + 180n + 13) = \frac{128}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2 + 27n + 3) = \frac{48}{\pi^2}.$$

I proved the three first formulas by the WZ-method in 2002 and 2003 and the last one in 2010.

Conjectured formulas

By the PSLQ algorithm we discovered the formulas

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (252n^2 + 63n + 5) = \frac{48}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{(1)_n^5} \frac{1}{7^{4n}} (1920n^2 + 304n + 15) = \frac{56\sqrt{7}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (1640n^2 + 278n + 15) = \frac{256\sqrt{3}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{80^{3n}} (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2}.$$

They remain unproved.

More conjectured formulas

In 2010 we discovered three more series

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} (-1)^n \left(\frac{3}{4}\right)^{6n} (1936n^2 + 549n + 45) = \frac{384}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \left(\frac{3}{5}\right)^{6n} (532n^2 + 126n + 9) = \frac{375}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{\phi}\right)^{3n} \left[(32 - \frac{216}{\phi}) n^2 + (18 - \frac{162}{\phi}) n + (3 - \frac{30}{\phi}) \right] = \frac{3}{\pi^2},$$

where ϕ is the fifth power of the golden ratio. This formula is the unique irrational example that I have found for $1/\pi^2$.

The second formula is joint with G. Almkvist.

Formulas for $1/\pi^3$ and $1/\pi^4$

B. Gourevitch (2002) found with PSLQ the formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7} \frac{1}{2^{6n}} (168n^3 + 76n^2 + 14n + 1) = \frac{32}{\pi^3},$$

and Jim Cullen (2010) found with PSLQ the formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^9} \frac{1}{2^{12n}} \times$$

$$(43680n^4 + 20632n^3 + 4340n^2 + 466n + 21) = \frac{2^{12}}{\pi^4}.$$

Are they provable by the WZ-method?.

The WZ-method

Let $G(n,k)$ be hypergeometric in its two symbols. The proof of

$$\sum_{n=0}^{\infty} G(n,k) = \text{Constant},$$

can be automatically (EKHAD) carried over by a computer.

H. Wilf and D. Zeilberger have discovered an algorithm that finds a rational function $C(n,k)$ called certificate, such that

$$F(n,k) = C(n,k)G(n,k), \qquad F(0,k) = 0,$$

$$G(n,k+1) - G(n,k) = F(n+1,k) - F(n,k) \qquad (WZ\text{–}pair).$$

Observe that if we sum for $n \geq 0$ the right side telescopes. Then apply Carlson's theorem.

Let $F(n,k)$ and $G(n,k)$ be the two hypergeometric functions of a WZ-pair, and suppose that in addition $F(0,k)=0$. If we define

$$F_{s,t}(n,k) = F(sn, k+tn), \qquad s \in \mathbb{Z} - \{0\}, \qquad t \in \mathbb{Z},$$

then $F_{s,t}(n,k)$ and $G_{s,t}(n,k)$ are also the functions of WZ-pairs satisfying $F_{s,t}(0,k)=0$ and in addition, we have

$$\sum_{n=0}^{\infty} G_{s,t}(n,k) = \sum_{n=0}^{\infty} G(n,k) = \text{Constant}.$$

So we have a chain of formulas with the same sum.

Bauer's series

In 1859 Bauer proved the formula

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}(4n+1) = \frac{2}{\pi}.$$

Generalization and Zeilberger's proof of Bauer's series:

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)_n^2 \left(\frac{1}{2}-k\right)_n}{(1+k)_n(1)_n^2}(4n+1) = \frac{2}{\pi}\frac{(1)_k}{\left(\frac{1}{2}\right)_k}.$$

Proof: The companion is

$$F(n,k) = (-1)^n \frac{\left(\frac{1}{2}\right)_n^2 \left(\frac{1}{2}-k\right)_n}{(1+k)_n(1)_n^2} \frac{\left(\frac{1}{2}\right)_k}{(1)_k} \frac{n^2}{2n-2k-1},$$

and we deduce the constant taking $k = 1/2$.

Some remarks

Write `with(SumTools[Hypergeometric]);` in a Maple session,

and let $$H(n,k) = (-1)^n \frac{\left(\frac{1}{2}\right)_n^2 \left(\frac{1}{2}-k\right)_n}{(1+k)_n (1)_n^2}.$$ Then, writing

```
degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

we see that the degree is $2 < 3$ (candidate). Then, if we write

```
coK2:=coeff(Zeilberger(H(n,k)*(n+b*k+c),k,n,K)[1],K,2);
coes:=coeffs(coK2,k); solve({coes},{b,c});
```

we get the solution $b = 0$, $c = 1/4$. Then, writing

```
Zeilberger(H(n,k)*(4*n+1),k,n,K)[1];
```

we get the output $(1+2k)K - (2+2k)$.

WZ-proofs of series for $1/\pi$

By the WZ-method, we get the identities:

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^2 (1+k)_n} (8n+2k+1) = \frac{2\sqrt{3}}{\pi} \left(\frac{4}{3}\right)^k \frac{(1)_k}{\left(\frac{1}{2}\right)_k}.$$

$F(n,k) \to F(n,k+n)$, leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n}3^n} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{4}+\frac{k}{2}\right)_n \left(\frac{3}{4}+\frac{k}{2}\right)_n \left(\frac{1}{2}+k\right)_n}{(1)_n^2 \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n}$$
$$\times \frac{(28n+3)(2n+1)+4k(9n+k+2)}{2n+k+1} = \frac{16\sqrt{3}}{3\pi} \cdot \left(\frac{4}{3}\right)^k \frac{(1)_k}{\left(\frac{1}{2}\right)_k}.$$

We have determined the values of the constants by taking $k = 1/2$.

More formulas in the chain

$F(n,k) \to F(n,k+2n)$ leads to

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{720n^3 + 804n^2 + 236n + 15}{(n+\frac{1}{3})(n+\frac{2}{3})} = \frac{128\sqrt{3}}{\pi}.$$

$F(n,k) \to F(2n,k-3n)$ leads to

$$\sum_{n=0}^{\infty} \frac{5^{5n}}{2^{6n}3^{5n}} \frac{\left(\frac{1}{10}\right)_n \left(\frac{3}{10}\right)_n \left(\frac{7}{10}\right)_n \left(\frac{9}{10}\right)_n}{(1)_n^3 \left(\frac{1}{2}\right)_n} \frac{2924n^2 + 1668n + 105}{n+\frac{1}{2}} = \frac{432\sqrt{3}}{\pi}.$$

By the WZ-method, we get the identities:

$$\sum_{n=0}^{\infty} \left(\frac{-1}{8}\right)^n \frac{\left(\frac{1}{2}+2k\right)_n \left(\frac{1}{2}\right)_n^2}{(1+k)_n^2 (1)_n}(6n+4k+1) = \frac{2\sqrt{2}}{\pi} \cdot \frac{(1)_k^2}{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}.$$

With the transformation $F(n,k) \to F(n,k+n)$, we get

$$\sum_{n=0}^{\infty} \left(\frac{-27}{512}\right)^n \frac{\left(\frac{1}{2}+2k\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n \left(1+\frac{k}{2}\right)_n (1+k)_n (1)_n}$$

$$\times \frac{(154n+15)(2n+1)+4k(66n+16k+19)}{2n+k+1} = \frac{32\sqrt{2}}{\pi} \cdot \frac{(1)_k^2}{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}.$$

Here, we have determined the constants taking $k \to +\infty$. Observe that $(k)_n \sim k^n$.

By the WZ-method, we get the identities:

$$\sum_{n=0}^{\infty} \left(\frac{-1}{16}\right)^n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+2k\right)_n \left(\frac{1}{3}+k\right)_n \left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n \left(1+\frac{k}{2}\right)_n (1+k)_n (1)_n}$$

$$\times \frac{(51n+7)(2n+1)+k(114n+36k+37)}{2n+k+1} = \frac{12\sqrt{3}}{\pi} \cdot \frac{(1)_k^2}{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}.$$

$$\sum_{n=0}^{\infty} \left(\frac{-9}{16}\right)^n \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+3k\right)_n \left(\frac{1}{3}+k\right)_n \left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}\right)_n (1)_n (1+k)_n (1+3k)_n}$$

$$\times \frac{(5n+1)(2n+1)+k(16n+6k+7)}{2n+1} = \frac{4\sqrt{3}}{3\pi} \cdot 4^k \cdot \frac{(1)_k^2}{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}.$$

We consider the following expression:

$$H(n,k) = \left(\frac{-1}{16}\right)^n \frac{\left(\frac{1}{2}+j_1\,k\right)_n \left(\frac{1}{2}+j_2\,k\right)_n \left(\frac{1}{3}+j_3\,k\right)_n \left(\frac{2}{3}+j_3\,k\right)_n}{\left(1+j_4\,\frac{k}{2}\right)_n \left(\frac{1}{2}+j_4\,\frac{k}{2}\right)_n (1+j_5\,k)_n (1)_n},$$

For most of the values of j_1, j_2, j_3, j_4 and j_5, we see (Maple):

```
with(SumTools[Hypergeometric]);
degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

is equal to 4, but for $j_1 = 0$, $j_2 = 2$, $j_3 = j_4 = j_5 = 1$, we see that

```
degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

is equal to 3. Hence, this is candidate.

With the candidate

$$H(n,k)=\left(\frac{-1}{16}\right)^n\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+2k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n\left(1+\frac{k}{2}\right)_n(1+k)_n(1)_n},$$

we calculate the numerical values of

$$A_k=\sum_{n=0}^{\infty}H(n,k)\frac{(51n+7)(2n+1)}{2n+k+1},$$

$$B_k=k\sum_{n=0}^{\infty}H(n,k)\frac{n}{2n+k+1},$$

$$C_k=k\sum_{n=0}^{\infty}H(n,k)\frac{1}{2n+k+1}.$$

and of $D=12\sqrt{3}/\pi$.

We see that we have to find the constants a_1, a_2 and a_3, such that

$$A_k + a_1 B_k + (a_2 k + a_3) C_k + bDf(k) = 0.$$

We find them using PSLQ to look for integer relations among

$$A_k, \qquad B_k, \qquad C_k, \qquad D.$$

We get

$$\begin{aligned} 3A_1 + 342B_1 + 219C_1 - 16Df(1) &= 0, \\ 105A_2 + 11970B_2 + 11445C_2 - 1024Df(2) &= 0, \\ 1155A_3 + 131670B_3 + 167475C_3 - 16384Df(3) &= 0. \end{aligned}$$

The solution is $a_1 = 114$, $a_2 = 36$ and $a_3 = 37$.

The combinatorial part of the WZ-pair is

$$B(n,k) = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+2k\right)_n \left(\frac{1}{3}+k\right)_n \left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n \left(1+\frac{k}{2}\right)_n (1+k)_n (1)_n} \cdot \frac{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(1)_k^2},$$

$$= \frac{\left(\frac{1}{3}+n\right)_k \left(\frac{2}{3}+n\right)_k \left(\frac{1}{4}+\frac{n}{2}\right)_k \left(\frac{3}{4}+\frac{n}{2}\right)_k}{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k (1+n)_k (1+2n)_k} \cdot \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3}.$$

And the WZ-pair is

$$G(n,k) = B(n,k) \left(-\frac{1}{16}\right)^n \frac{(51n+7)(2n+1)+k(114n+36k+37)}{2n+k+1},$$

$$F(n,k) = B(n,k) \left(-\frac{1}{16}\right)^n \frac{9n(-6n^2-30nk-13n+7k+3)}{(3k+1)(3k+2)}.$$

Observe how we guess the denominators of the rational parts.

Another example

We have

$$\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}+k\right)_n\left(\frac{1}{4}-k\right)_n}{(1)_n^2(1+k)_n\left(\frac{1}{4}+k\right)_n} \times \frac{(3+20n)(4n+1)+4k(12n+1)}{4n+4k+1} = \frac{8}{\pi}\frac{\left(\frac{1}{4}\right)_k(1)_k}{\left(\frac{3}{4}\right)_k\left(\frac{1}{2}\right)_k}.$$

And

$$F(n,k) = \left(-\frac{1}{4}\right)^n \frac{\left(\frac{1}{4}\right)_k\left(\frac{1}{2}\right)_k\left(\frac{3}{4}+n\right)_k}{(1+n)_k\left(\frac{3}{4}-n\right)_k\left(\frac{1}{4}+n\right)_k}\cdot\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(1)_n^3} \times \frac{64n^2(4n-1)}{(4n-4k-3)(4n+4k+1)}.$$

Observe how we guess the denominators of the rational parts.

From $s = 1/2$ to $s = 1/4$

We have not found a WZ-pair to prove the formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (-1)^n \left(\frac{16}{63}\right)^{2n} (65n+8) = \frac{9\sqrt{7}}{\pi},$$

but we can relate it to a formula proved by the WZ-method. Let

$$A(n,k) = 3\left(\frac{64}{63}\right)^k \frac{(-k)_n \left(\frac{1}{2}\right)_n^2}{\left(\frac{1}{2}-k\right)_n^2 (1)_n} \left(\frac{1}{64}\right)^n (42n+5),$$

$$B(n,k) = \frac{(-k)_n \left(\frac{-k}{2}\right)_n \left(\frac{1}{2}-\frac{k}{2}\right)_n}{\left(\frac{1}{2}-k\right)_n^2 (1)_n} (-1)^n \left(\frac{16}{63}\right)^{2n} (130n-2k+15).$$

From a Whipple's formula we can deduce that

$$\sum_{n=0}^{\infty} A(n,k) = \sum_{n=0}^{\infty} B(n,k) \quad \forall k \in \mathbb{C}.$$

Automatic proof

$$\text{Let} \quad a(k) = \sum_{n=0}^{\infty} A(n,k), \quad b(k) = \sum_{n=0}^{\infty} B(n,k).$$

We can prove that $a(k) = b(k)$ automatically using `Zeilberger`:

```
with(SumTools[Hypergeometric]);
Zeilberger(A(n,k),k,n,K)[1];
Zeilberger(B(n,k),k,n,K)[1];
```

We see that $a(k)$ and $b(k)$ satisfy the same third order recurrent equation, and due to $(-k)_n$, we can directly check that

$$a(0) = b(0), \quad a(1) = b(1), \quad a(2) = b(2).$$

Hence $a(k) = b(k)$ for all integers, which imply (Carlson's Thm.) that $a(k) = b(k) \quad \forall k \in \mathbb{C}$. Replacing $k = -1/2$ we are done.

From $s = 1/2$ to $s = 1/6$

Prove that:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left(\frac{2}{11}\right)^{3n} (126n+10) = \frac{11\sqrt{33}}{2\pi}.$$

With

```
Zeilberger(f(n,k),k,n,K)[1];
```

we can automatically prove that

$$11\left(\frac{32}{33}\right)^{3k} \sum_{n=0}^{\infty} \frac{(-3k)_n \left(\frac{1}{3}-k\right)_n \left(\frac{1}{6}-2k\right)_n}{\left(\frac{2}{3}-2k\right)_n \left(\frac{1}{3}-4k\right)_n (1)_n} \left(\frac{-1}{8}\right)^n (6n+1)$$

$$= \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{3}-k\right)_n \left(\frac{2}{3}-k\right)_n}{\left(\frac{5}{6}-k\right)_n \left(\frac{2}{3}-2k\right)_n (1)_n} \left(\frac{2}{11}\right)^{3n} (126n+6k+11).$$

Here take $k = -1/6$ and we are done.

D. Zeilberger wrote the Maple package `twoFone`, which found automatically many nice formulas, like for example

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}-k\right)_n \left(\frac{1}{4}-3k\right)_n}{(1+2k)_n (1)_n} (9-4\sqrt{5})^n = C_1 \frac{2^{8k}}{5^{2k}(5+2\sqrt{5})^k} \frac{(1)_k \left(\frac{1}{2}\right)_k}{\left(\frac{11}{20}\right)_k \left(\frac{19}{20}\right)_k}.$$

Multiplying (inside the series) for $n+bk+c$, we determine b and c forcing the coefficient of K^2 to be 0. That is, writing

```
coK2:=coeff(Zeilberger(k,n,K)[1],K,2);
coes:=coeffs(coK2,k);
solve({coes},{b,c});
```

and we obtain the complementary formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}-k\right)_n \left(\frac{1}{4}-3k\right)_n}{(1+2k)_n (1)_n} (9-4\sqrt{5})^n \left[40n+20(\sqrt{5}-1)k+5-\sqrt{5}\right]$$

$$= C_2 \frac{2^{8k}}{5^{2k}(5+2\sqrt{5})^k} \frac{(1)_k \left(\frac{1}{2}\right)_k}{\left(\frac{3}{20}\right)_k \left(\frac{7}{20}\right)_k}, \qquad C_1 C_2 = \frac{2\sqrt{10+5\sqrt{5}}}{\pi}.$$

Substituting $k=0$, and multiplying both series, we obtain

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n^2}{(1)_n^2} (9-4\sqrt{5})^n \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n^2}{(1)_n^2} (9-4\sqrt{5})^n (40n+5-\sqrt{5}) = C_1 C_2.$$

Finally, using Clausen formula, the product transforms into

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (9-4\sqrt{5})^n (20n+5-\sqrt{5}) = \frac{2\sqrt{10+5\sqrt{5}}}{\pi}.$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^3 (1+k)_n^2} (120n^2+84kn+34n+10k+3) = \frac{32}{\pi^2} \frac{(1)_k^2}{\left(\frac{1}{2}\right)_k^2}.$$

For $k=0$ we have

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} (120n^2+34n+3) = \frac{32}{\pi^2},$$

and if we let $k \to \infty$, we recover the Ramanujan series

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) = \frac{16}{\pi}.$$

Observe that $(k)_n \sim k^n$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n (1+k)_n^4} (20n^2+8n+1+24kn+8k^2+4k) = \frac{8}{\pi^2} \frac{(1)_k^4}{\left(\frac{1}{2}\right)_k^4}.$$

For $k = 0$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} (20n^2 + 8n + 1) = \frac{8}{\pi^2}.$$

With the transformation $F(n,k) \to F(n,k+n)$, we get

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (820n^2 + 180n + 13) = \frac{128}{\pi^2},$$

which gives 3 digits per term.

We have

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}+\frac{k}{3}\right)_n \left(\frac{2}{3}+\frac{k}{3}\right)_n \left(1+\frac{k}{3}\right)_n}{(1)_n^3 (1+k)_n^3} \left(\frac{3}{4}\right)^{3n} \times \frac{(74n^2+27n+3)n + k(108n^2+42kn+24n+5k+1)}{n+\frac{k}{3}}$$

$$= \frac{48}{\pi^2} \frac{(1)_k^2}{\left(\frac{1}{2}\right)_k^2}, \quad \text{(we get the constant taking the limit as } k \to \infty).$$

Then, taking $k=0$, we get

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2+27n+3) = \frac{48}{\pi^2}.$$

I proved this formula in (2010).

WZ-proof of another formula by Ramanujan

In his first letter to Hardy, Ramanujan sent the following formula:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} (-1)^n (4n+1) = \frac{2}{\Gamma^4\left(\frac{3}{4}\right)}.$$

For $B(n,k) = \dfrac{\left(\frac{1}{2}\right)_n^2 \left(\frac{1}{2}+k\right)_n^2 \left(\frac{1}{2}-k\right)_n}{(1)_n^2 (1+k)_n^2 (1+2k)_n} (-1)^n$, we get that

```
degree(Zeilberger(B(n,k),k,n,K)[1],K)
```

is equal to 4, so this binomial part is a candidate. We find:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \left(\frac{1}{2}+k\right)_n^2 \left(\frac{1}{2}-k\right)_n}{(1)_n^2 (1+k)_n^2 (1+2k)_n} (-1)^n (4n+2k+1) = \frac{2}{\Gamma^4\left(\frac{3}{4}\right)} \frac{(1)_k^3}{\left(\frac{1}{2}\right)_k \left(\frac{3}{4}\right)_k^2},$$

which proves the Ramanujan series.

A complementary formula

We have found the following related series

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5}(4n+1)(8n^2+4n+1) = \frac{8\,\Gamma^4\left(\frac{3}{4}\right)}{\pi^4},$$

and the WZ-proof:

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)_n^2\left(\frac{1}{2}+k\right)_n^2\left(\frac{1}{2}-k\right)_n}{(1)_n^2(1+k)_n^2(1+2k)_n}\Big[(4n+1)(8n^2+4n+1)$$

$$+k(24n^2+8kn+8n+1)\Big] = \frac{8\,\Gamma^4\left(\frac{3}{4}\right)}{\pi^4}\,\frac{(1)_k^3}{\left(\frac{1}{2}\right)_k\left(\frac{1}{4}\right)_k^2}.$$

Hence, we have

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5}(4n+1)\cdot\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5}(4n+1)(8n^2+4n+1) = \frac{16}{\pi^4}.$$

Supercongruences

W. Zudilin used the WZ-method to prove *p-adic analogues* for some Ramanujan-type series for $1/\pi$ and $1/\pi^2$. For example:

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (20n+3) \frac{(-1)^n}{2^{2n}} \equiv 3(-1)^{\frac{p-1}{2}} p \pmod{p^3},$$

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (120n^2+34n+3) \frac{1}{2^{4n}} \equiv 3p^2 \pmod{p^5},$$

where p is an odd prime. I have observed that there are also p-adic analogues for the *product of complementary series*:

$$\sum_{n=0}^{p-1} (-1)^n \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} (4n+1) \cdot \sum_{n=0}^{p-1} (-1)^n \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} (4n+1)(8n^2+4n+1)$$

$$\equiv p^4 \pmod{p^6}, \quad \text{where p is an odd prime.}$$

Curious repetitions of special values of z. Part 1

Observe that these three series have the same value of z:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{1}{7^{4n}} (40n+3) = \frac{49\sqrt{3}}{9\pi},$$

proved with modular equations.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \frac{1}{7^{4n}} \frac{1920n^2+1072n+55}{2n+1} = \frac{196\sqrt{7}}{3\pi},$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{(1)_n^5} \frac{1}{7^{4n}} (1920n^2+304n+15) = \frac{56\sqrt{7}}{\pi^2},$$

unproved.

Observe that these two unproved series have the same value of z:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \left(\frac{3}{5}\right)^{6n} (532n^2 + 126n + 9) = \frac{375}{\pi^2},$$

(joint with G. Almkvist), and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \left(\frac{3}{5}\right)^{6n} \frac{133n^2 + 79n + 6}{2n+1} = \frac{625}{32\pi},$$

which I found recently by using the PSLQ algorithm.

Curious repetitions of special values of z. Part 3

Observe that these two series have the same value of z:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{(-1)^n}{48^n} (28n+3) = \frac{16\sqrt{3}}{3\pi},$$

proved by the modular theory and also by the WZ-method, and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (252n^2+63n+5) = \frac{48}{\pi^2},$$

unproved.

Curious repetitions of special values of z. Part 4

Observe that these two series have the same value of z:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \frac{(-1)^n}{80^{3n}} (5418n + 263) = \frac{640\sqrt{15}}{3\pi},$$

proved by the modular theory, and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{80^{3n}} (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2},$$

unproved.

Possible explanations

1. Similar WZ-pairs.
2. Cases $k = 0$ and limit as $k \to +\infty$ of the same formula.
3. Identities with a free parameter k.
4. Unknown transformations.

Thank you