Polynomial Approximation of Computationally-Hard Sequences

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Abstract

The question whether an m-tuple $(x_1, \ldots, x_m) \in \mathbb{Z}_{\geq 0}^m$ is in $(a^1, \ldots, a^m)$, where the a^i are given integer sequences, can sometimes be decided *efficiently* (in polynomial time). More often, the answer is unknown, the best known algorithms being exponential. We present a polynomial approximate algorithm for deciding this question for some sequences a_i. Specifically, we consider the complementary sequences $a_n = \text{mex}\{a_i, b_i : 0 \leq i < n\}$, $b_n = a_n + \lfloor n/k \rfloor$ (sequences A102528/9 in Sloane's encyclopedia for $k = 2$). Using Fekete's Lemma we show that the polynomially computable sequences $s_n = \lfloor n\alpha \rfloor$, $t_n = \lfloor n\beta \rfloor$ where $\alpha = (\sqrt{17} + 3)/4$, $\beta = \alpha + 1/2$ are very good approximations, namely, $s_{n-1} \leq a_n \leq s_n$, $t_{n-1} \leq b_n \leq t_n$ for all $n \geq 1$ $(k = 2)$. We conjecture that the percentage of n for which $a_n = s_n - 1$ is about 73%, $a_n = s_n$, 19%, $a_n = s_n - 2$, 8%. Similar results for b_n, t_n. Analogous results for every fixed $k > 1$. Existence of a limiting distribution, with one of the percentages dominant, would lead to first probabilistic algorithms for determining the winning positions of certain impartial games, where the $(a^1, \ldots, a^m)$ are the second player winning positions.

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