Nonlinear Dynamics in a Time of High Dimensional Nonlinear Data

Konstantin Mischaikow

Dept. of Mathematics/BioMaPS Institute
Rutgers
mischaik@math.rutgers.edu

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Philosophy

We live in a world where the ambient space within which data lies is often extremely high (infinite) dimensional.

Data that can be accessed, processed, and used in a timely fashion must lie on much lower dimensional spaces, but these spaces and/or the actions on these spaces can be highly nonlinear.

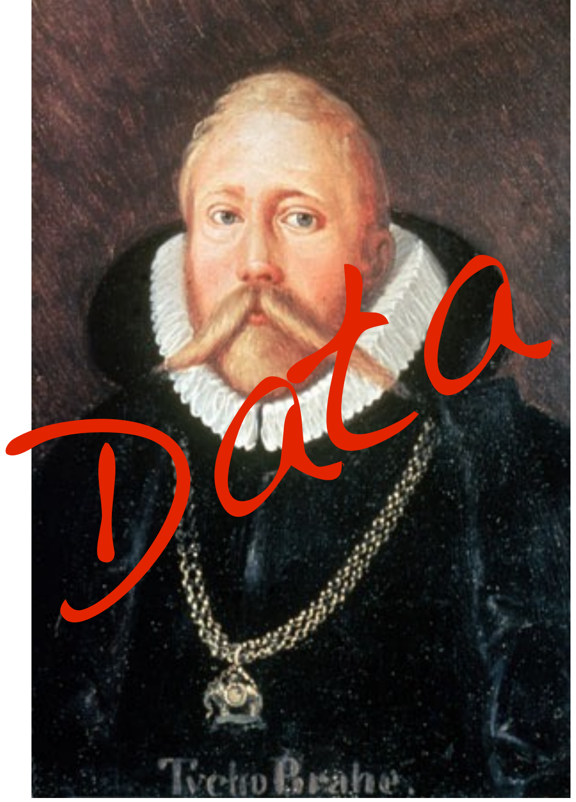
Algebraic topology is the mathematical tool for studying global nonlinear structures and maps. Homology is the most computationally accessible part of algebraic topology.

Lecture 1

Nonlinear Dynamics:
A
combinatorial
algebraic topological
paradigm

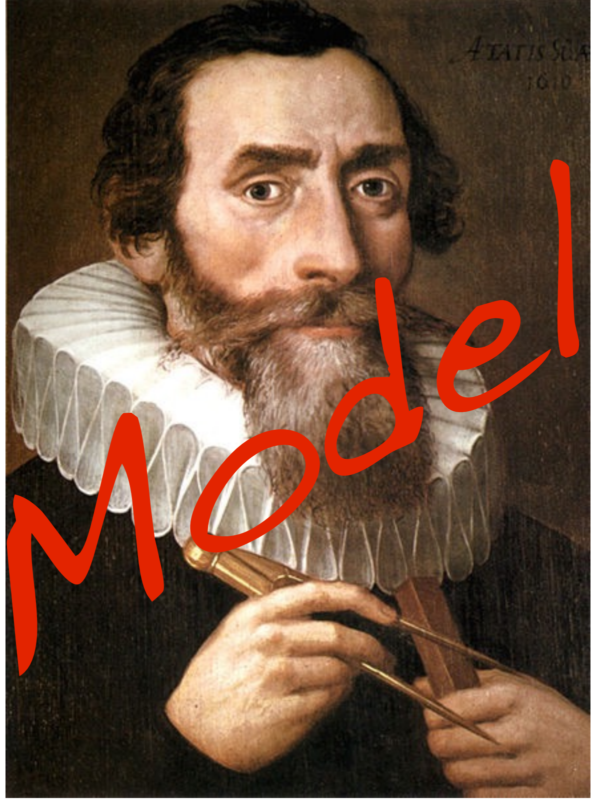
Motivation: why do we need a new approach to dynamics?

Differential Equations
and
Exact Solutions



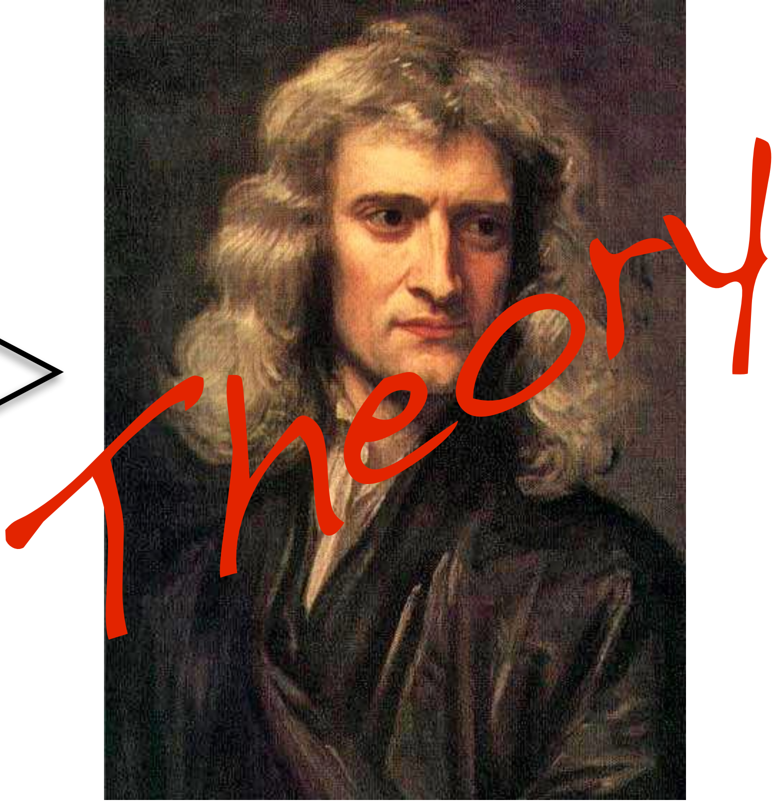
Tycho Brahe
1546-1601

→



Johannes Kepler
1571-1630

→



Isaac Newton
1643-1727

The 2 Body Problem:
$$m_i\ddot{q}_i = \sum_{i=1}^{2} \frac{Gm_1m_2(q_j - q_i)}{\|q_i - q_j\|^3} \qquad i = 1, 2$$

1. G, q and m are well defined and can be measured accurately.
2. Applications of this equation involve approximations, *point masses*, *2-body vs. n-body*, etc.
3. We accept the quantitative predictions.

Perihelion of Mercury: a serious challenge to Newtonian gravity

Sources of the precession of perihelion for Mercury

Amount (arcsec/Julian century)	**Cause**
5028.83 ±0.04	Coordinate (due to the precession of the equinoxes)
530	Gravitational tugs of the other planets
0.0254	Oblateness of the Sun (quadrupole moment)
42.98 ±0.04	General relativity
5603.24	Total
5599.7	Observed
–3.54 (–0.0632%)	Discrepancy

Urbain Le Verrier 1859 - rate of precession disagree's with Newtonian prediction by 38 arcsec/Julian century

Planet Vulcan was proposed to exist between Sun and Mercury

wikipedia

GROWTH OF POPULATION OF PARAMECIUM CAUDATUM

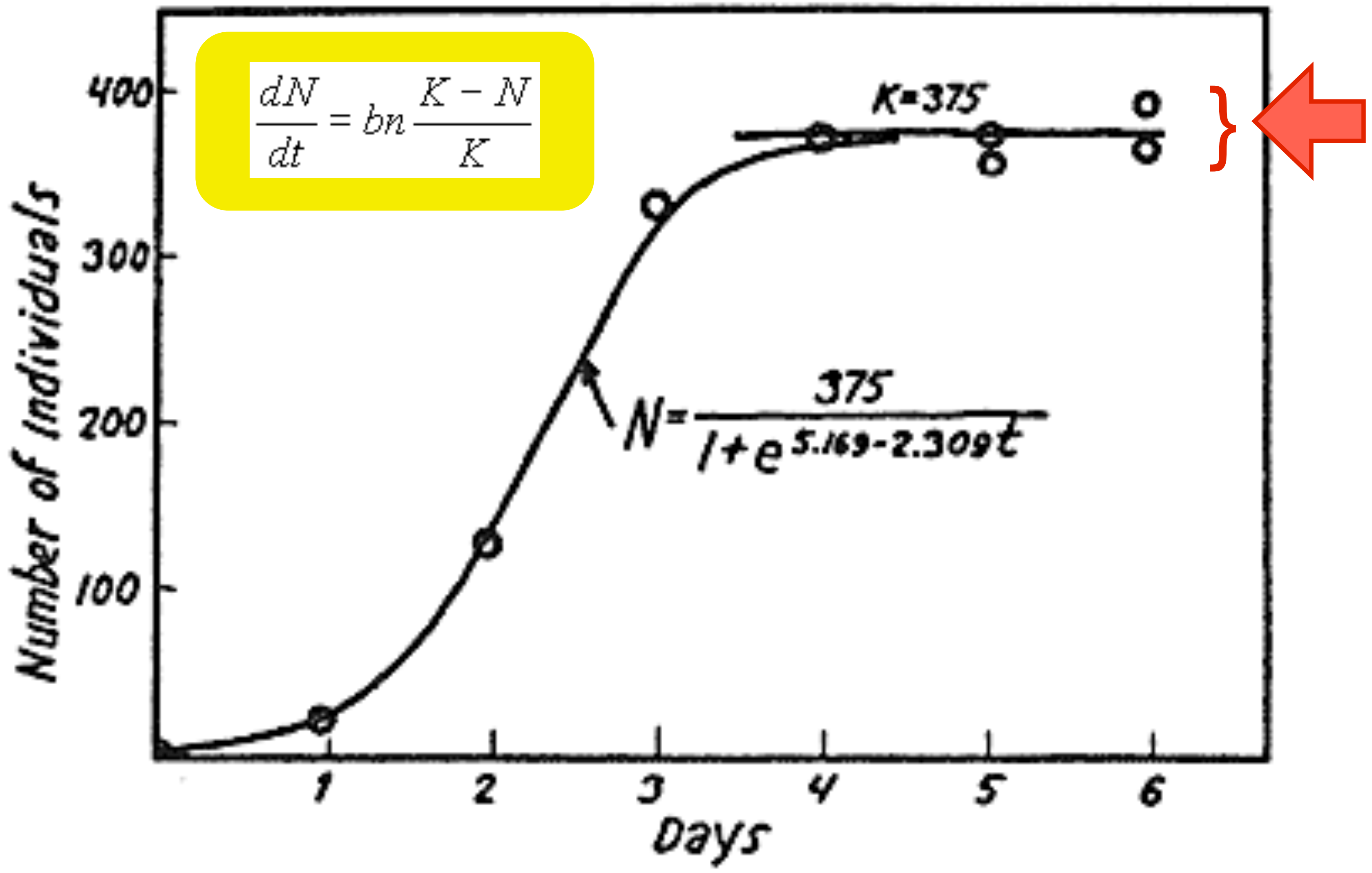
A mathematical model: $\dot{x} = \frac{dx}{dt} = rx(\kappa - x)$

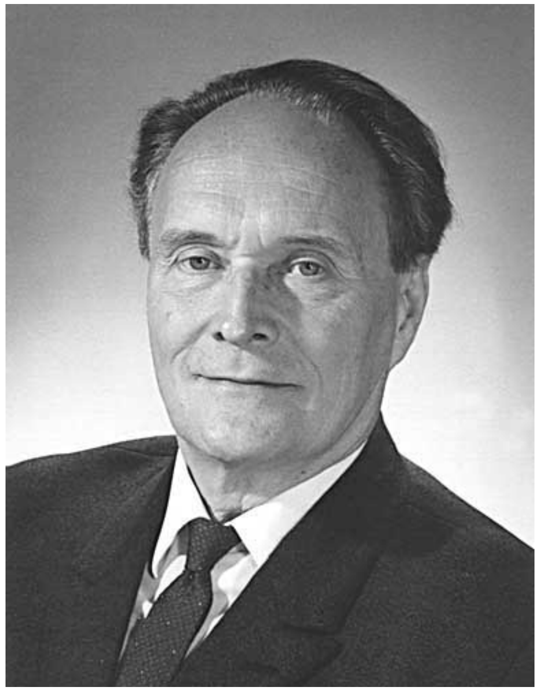
(logistic equation)

r is the birth rate κ is the carrying capacity of environment

Explicit solution given initial population x_0 at time $t = 0$ is

$$x(t) = \frac{x_0 \kappa e^{r\kappa t}}{\kappa - x_0 + x_0 e^{r\kappa t}}$$





Georgyi Frantsevitch Gause, 1910-1986

http://www.ggause.com/Contgau.htm

1. What do we mean by b and K? How precisely can we measure them?

2. My interpretation of this model is that it represents a Platonic ideal as opposed to a physical reality.

How important is mathematical theory (I)?

THE AEROSPACE CORPORATION

Precision Modeling for Orbit Determination

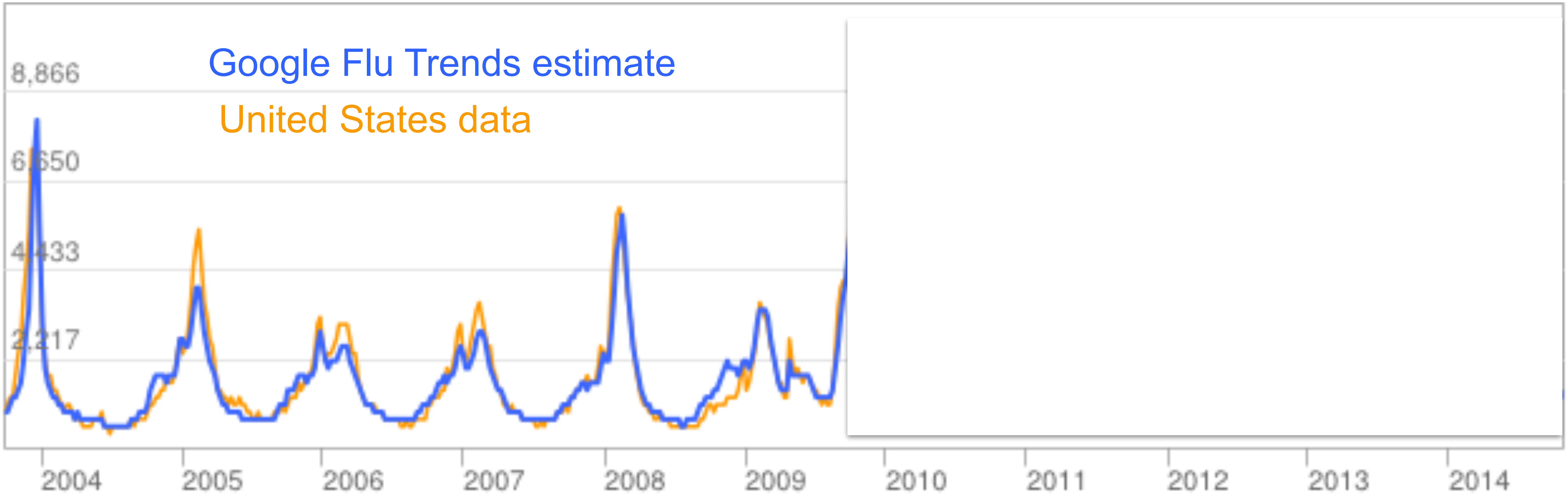
High overhead, more than 20,000 kilometers above Earth, GPS satellites race by at speeds approaching 3800 meters per second. The movements of these spacecraft are generally described by the laws of planetary motion developed by Johannes Kepler almost 400 years ago—but they are by no means certain or simple. Each satellite must contend with diverse forces that constantly nudge and pull it from its desired orbit. Yet in spite of this, the positions of GPS satellites must be known at all times with exceptional accuracy. Modeling these orbits is a complex affair. Here are just a few of the many issues that must be considered.

http://www.aero.org/publications/crosslink/summer2002/04.html

How important is mathematical theory (II)?

United States Flu Activity

Influenza estimate



http://www.google.org/flutrends/about/how.html

Dynamical Systems:
Qualitative Theory
of
Differential Equations

Jules Henri Poincare
1854-1912

The 3-body Problem ≈1890

1. Closed form solutions need not exist.
2. Chaotic dynamics exists.

⇒ Understanding the solution of an single initial value problem is not sufficient.

3. Invented Algebraic Topology.

Need to consider all solutions: $\dfrac{dx}{dt} = g(x), \quad x \in \mathbb{R}^n$

Flow: $\varphi \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$$(t, x) \mapsto \varphi(t, x)$$

time

initial
condition

value of solution
at time t

Map: $f \colon \mathbb{R}^n \to \mathbb{R}^n$

$$x \mapsto f(x) := \varphi(\tau, x)$$

$\tau > 0$ is a fixed time.

$S \subset \mathbb{R}^n$ is an invariant set if $\varphi(t, S) = S$ for all times t.

Steven Smale
1930-

$f\colon X \times \Lambda \to X$ differentiable

$$(x, \lambda) \mapsto f_\lambda(x)$$

The objects of interest:

A set $S \subset X$ is invariant if $f(S) = S$.

Examples: equilibria, periodic orbits, heteroclinic orbits, strange attractors

The equivalence relation:

Two maps $f\colon X \to X$ and $g\colon Y \to Y$ are topologically conjugate if there exists a homeomorphism $h\colon X \to Y$ such that $h \circ f = g \circ h$.

The places of change:

$\lambda_0 \in \Lambda$ is a bifurcation point if for any neighborhood U of λ_0 there exists $\lambda_1 \in U$ such that f_{λ_0} is not conjugate to f_{λ_1}

Given a family of dynamical systems (differential equations), what *types of dynamical structures* does one expect to see *typically*?

Modeling via Data

Anthony R. Ives, Árni Einarsson,
Vincent A. A. Jansen and Arnthor
Gardarsson,
High-amplitude fluctuations and
alternative dynamical states of
midges in Lake Myvatn,
Nature, vol. 452 (7183) pp. 84-87

Biological background of Myvatn ecosystem:

Lake Myvatn is a shallow, naturally eutrophic lake in northern Iceland.

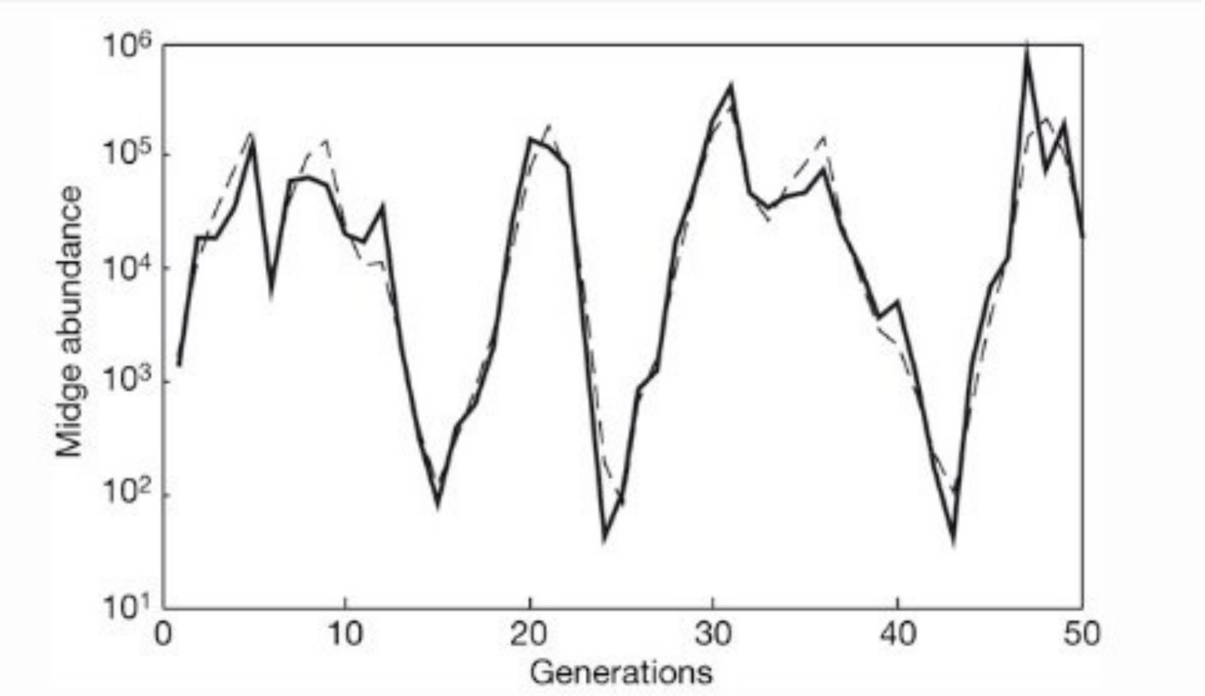
Eutrophic: Having waters rich in mineral and organic nutrients that promote a proliferation of plant life, especially algae, which reduces the dissolved oxygen content and often causes the extinction of other organisms.

Midges, *Tanytarsus gracilentus*, are the dominant herbivore/detritivore.

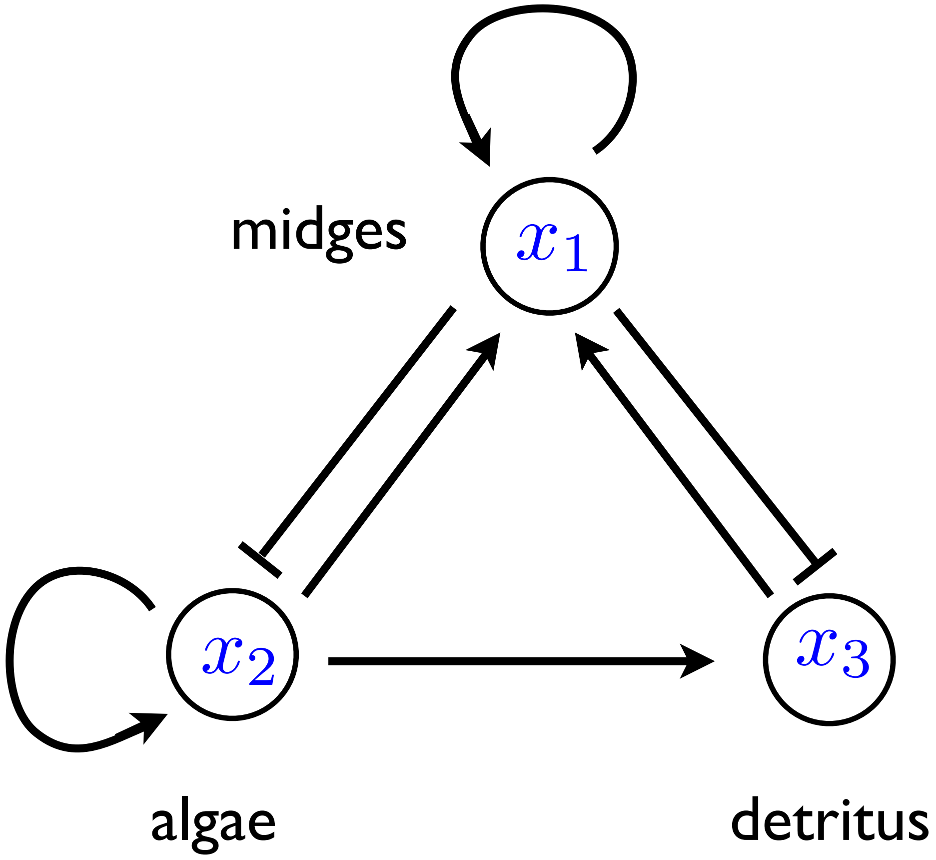
1. Two non-overlapping generations per year (first in May, second in late July early August)

2. Statistical evidence suggests that fluctuations in midge populations are driven by consumer–resource interactions, with midges being the consumers and algae/detritus the resources, as opposed to predator–prey interactions with midges being the prey.

3. Population levels of midges have been collected since 1977.



Making a mathematical model:



Non-overlapping generations ⇒ Map

$$\begin{aligned}
x_1(t+1) &= f_1(x_1(t), x_2(t), x_3(t), \lambda) \\
x_2(t+1) &= f_2(x_1(t), x_2(t), x_3(t), \lambda) \\
x_3(t+1) &= f_3(x_1(t), x_2(t), x_3(t), \lambda)
\end{aligned}$$

Gompertz log-linear model.

$$x_i(t+1) = \exp\left(\sum_{j=1}^{3} \lambda_{ij} \ln x_j\right)$$

$$\lambda = \begin{bmatrix} + & + & + \\ - & + & 0 \\ - & + & 0 \end{bmatrix}$$

Lotka-Volterra model

$$x_i(t+1) = r_i x_i(t) \exp\left(1 + \sum_{j=1}^{3} b_{ij} x_j(t)\right)$$

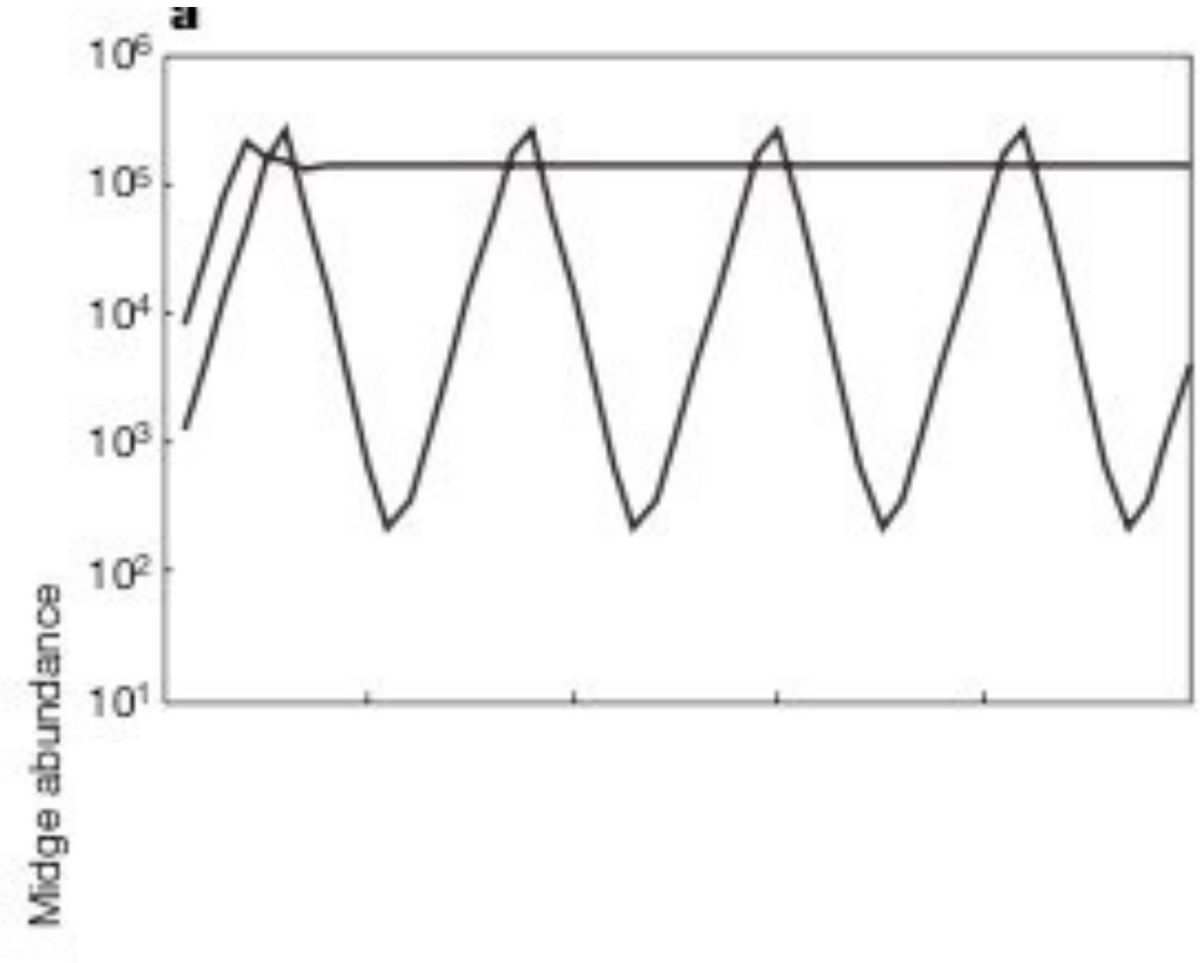
Proposed model (IEJG):

$$\begin{aligned} x_1(t+1) &= r_1 x_1(t)\left(1+\frac{x_1(t)}{x_2(t)+px_3(t)}\right)^{-q} \\ x_2(t+1) &= r_2\frac{x_2(t)}{1+x_2(t)}-\frac{x_2(t)}{x_2(t)+px_3(t)}x_1(t+1)+c \\ x_3(t+1) &= dx_3(t)+x_2(t)-\frac{px_3(t)}{x_2(t)+px_3(t)}x_1(t+1)+c \end{aligned}$$

- r_1 is the intrinsic population growth rate for midges
- r_2 is the intrinsic population growth rate for algae
- q density dependence parameter
- p quality of detritus for midges in relation to algae
- c is influx rate of algae from environment
- d retention rate of detritus in environment

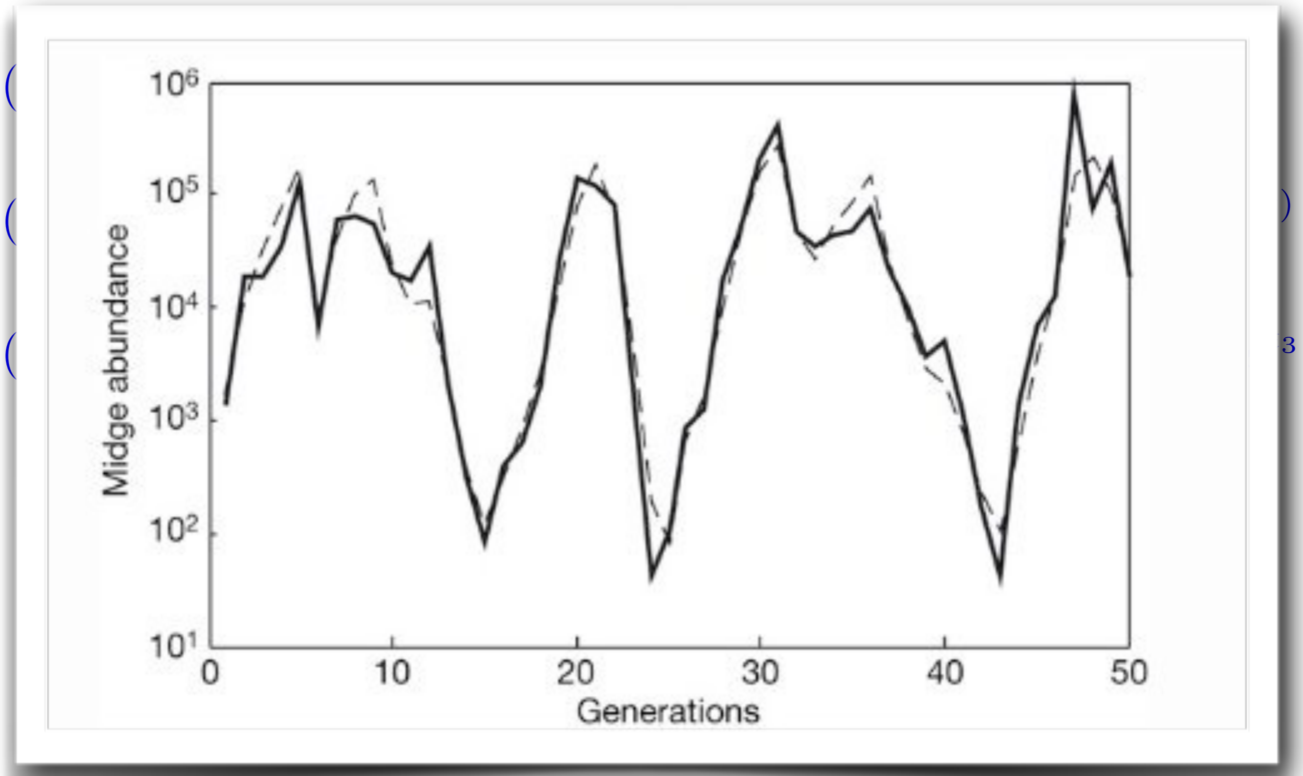
How can we justify this model over others?

Fact: There exist parameter values of IEJG model which exhibit multiple basins of attraction. This is not true for other models.



$$\begin{aligned}
x_1(t+1) &= r_1 x_1(t)\left(1+\frac{x_1(t)}{x_2(t)+px_3(t)}\right)^{-q} \\
x_2(t+1) &= r_2\frac{x_2(t)}{1+x_2(t)} - \frac{x_2(t)}{x_2(t)+px_3(t)}x_1(t+1)+c \\
x_3(t+1) &= dx_3(t)+x_2(t) - \frac{px_3(t)}{x_2(t)+px_3(t)}x_1(t+1)+c
\end{aligned}$$

$$\begin{aligned}
r_1 &= 3.873, & r_2 &= 11.746, \\
\boxed{c = 10^{-6.435},} & & d &= 0.5517, \\
p &= 0.06659, & q &= 0.9026
\end{aligned}$$

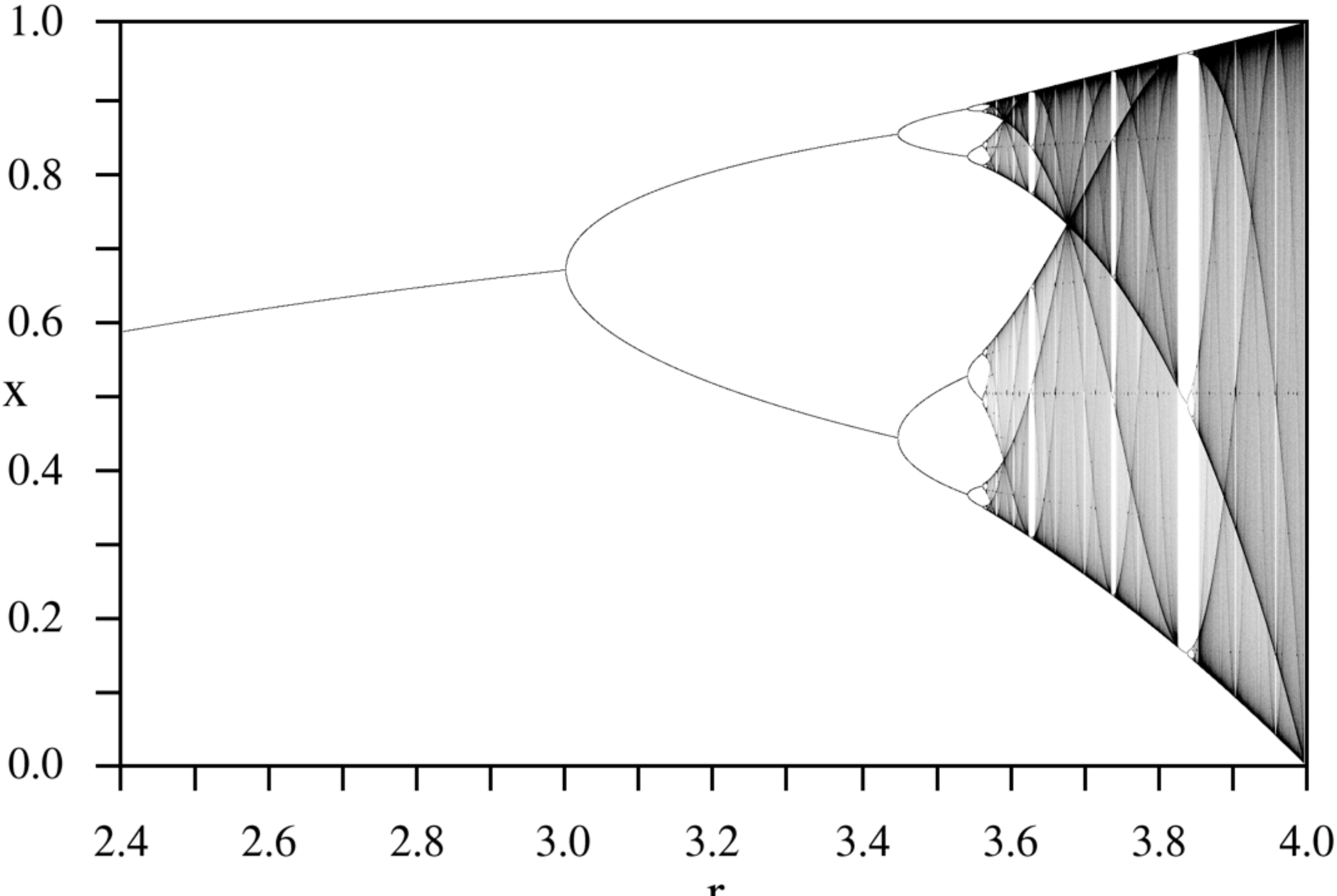


A striking biological conclusion from the model is the sensitivity of the amplitude of midge fluctuations to very small amounts of resource input; the resource input sets the lower boundary of midge abundance and hence the severity of population crashes. Thus, even though resource input might be six orders of magnitude less than the abundance of resources in the lake in most years, this vanishingly small source of resources is nevertheless critical in setting the depth of the midge population nadir and the subsequent rate of recovery. This sensitivity to resource subsidies might explain changes in midge dynamics that have apparently occurred over the last decades. Although Myvatn has supported a local charr (salmonid) fishery for centuries, this fishery collapsed in the 1980s, coincident with particularly severe midge population crashes. Over the same period, waterbird reproduction in Myvatn was also greatly reduced during the crash years. These changes might have been caused by dredging in one of the two basins in the lake that started in 1967 to extract diatomite from the sediment. Hydrological studies indicate that dredging produces depressions that act as effective traps of organic particles, hence reducing algae and detritus inputs to the midge habitat. Our model predicts that even a slight reduction in subsidies can markedly increase the magnitude of midge fluctuations. Such slight environmental changes can then have seriously negative consequences for fish and bird populations.

THE CHALLENGE OF BIFURCATIONS

$$x(t+1) = rx(t)(1 - x(t))$$

logistic map



Assume the logistic map

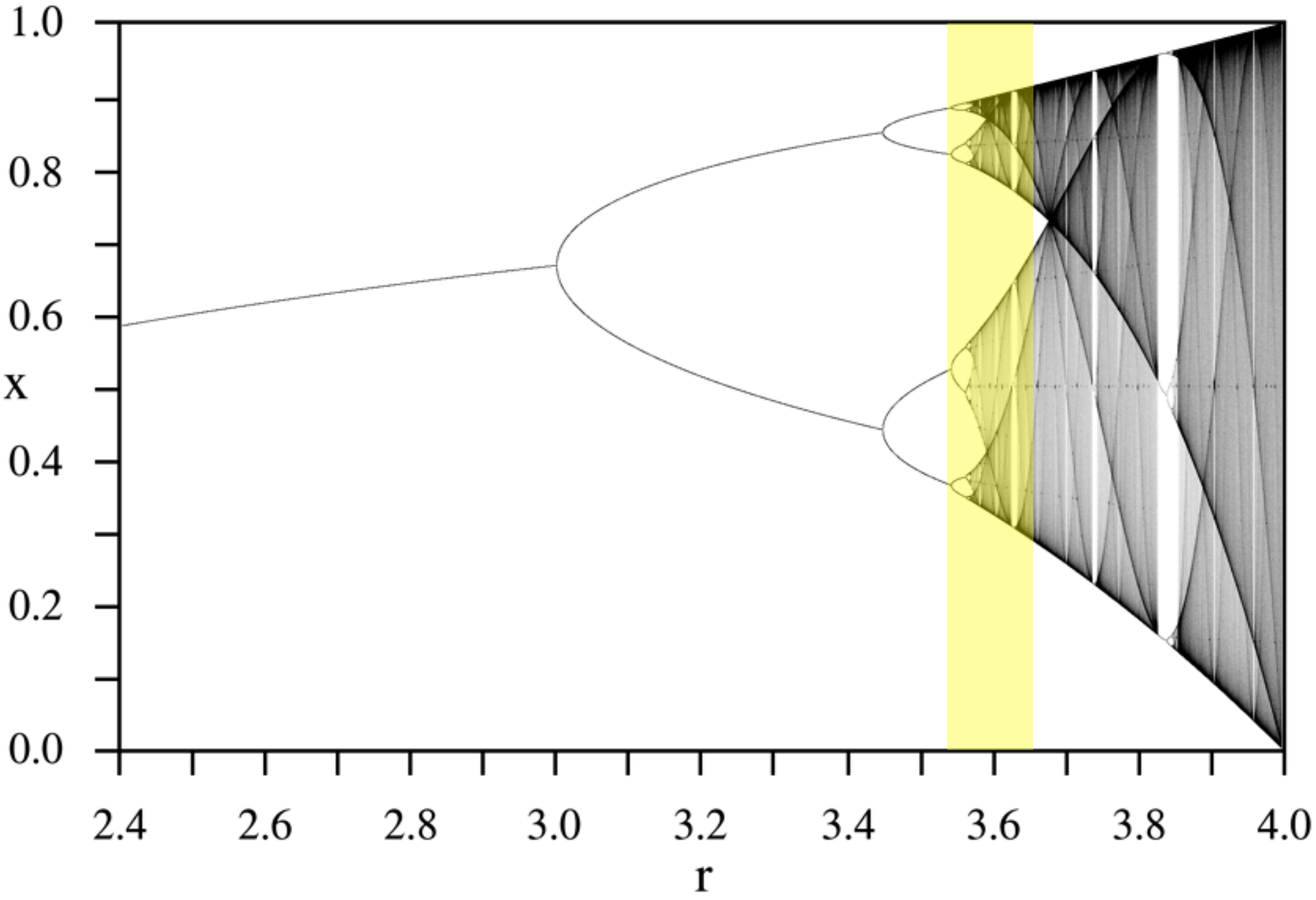
$$f(x) = rx(1-x)$$

is a perfect model for population growth of an insect.

Assume I can perform perfect numerical simulations of the model.

It is still possible that with high probability the conclusions will be wrong!

Assume the field biologist can measure birth rate to within one decimal place.



Any computation probably suggests the wrong dynamics!

Smale, Newhouse, ...

Invariant sets can change on parameter sets of positive measure

SUMMARY REMARKS

Technology is changing science:

1. We can collect enormous quantities of data. Scientists are relying more on data sets to characterize phenomena as opposed to mathematical theories.

2. We can perform large scale numerical studies of nonlinear systems.

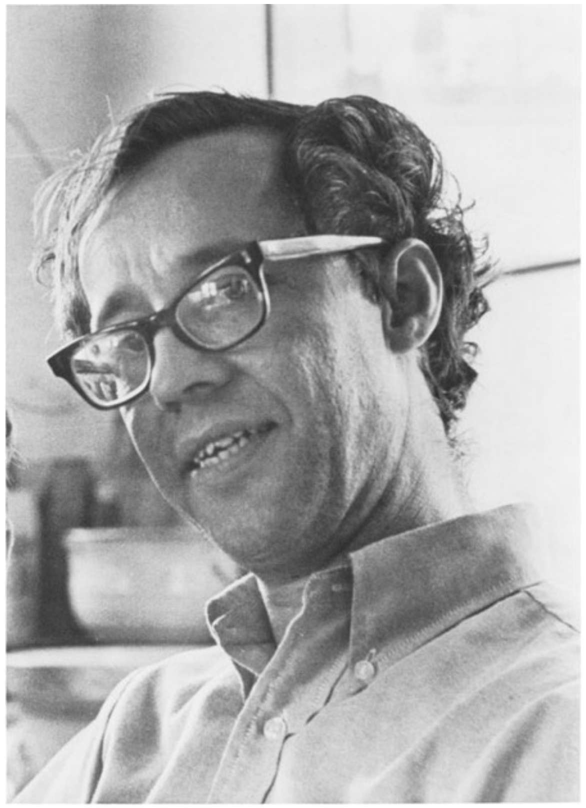
3. We are studying multiscale systems (biology, economics, medicine, for which nonlinearities and parameters are not known.

Remarks:

1. There are too many invariant sets so we require a new concept for `solving' differential equations/dynamical systems.

2. Should be compatible with Classical Theory, i.e. we can relate back to invariant sets.

SOME SIMPLE OBSERVATIONS: TOWARDS A NEW FRAMEWORK

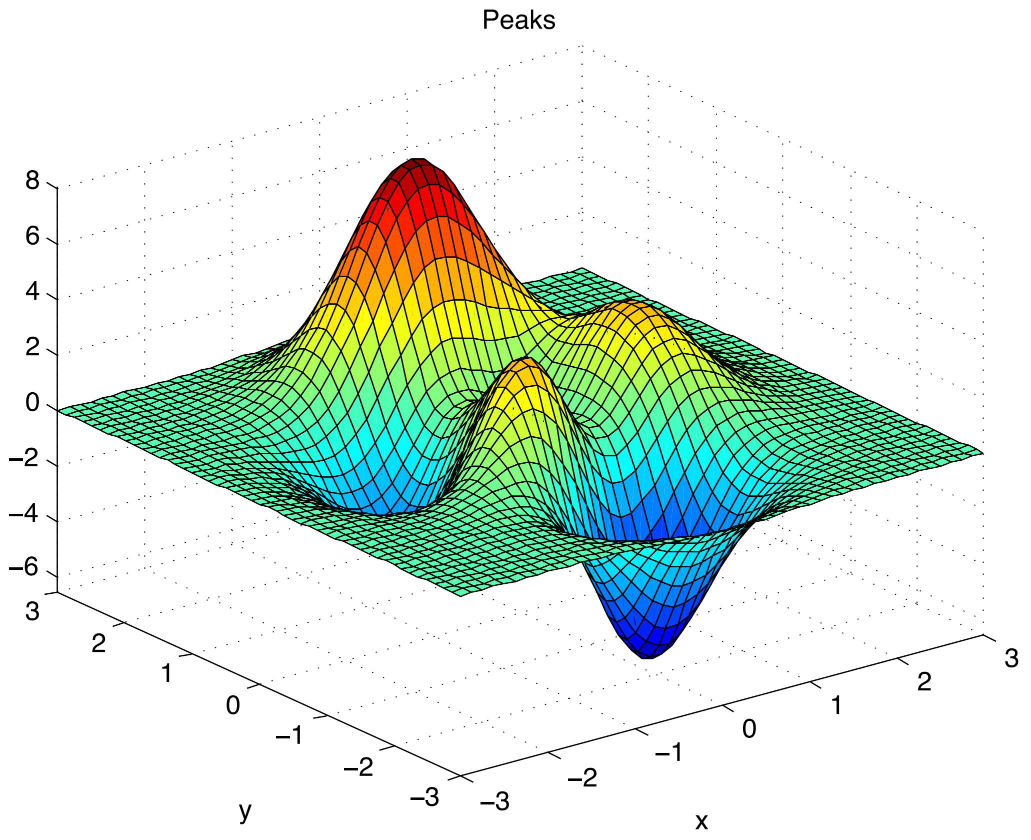


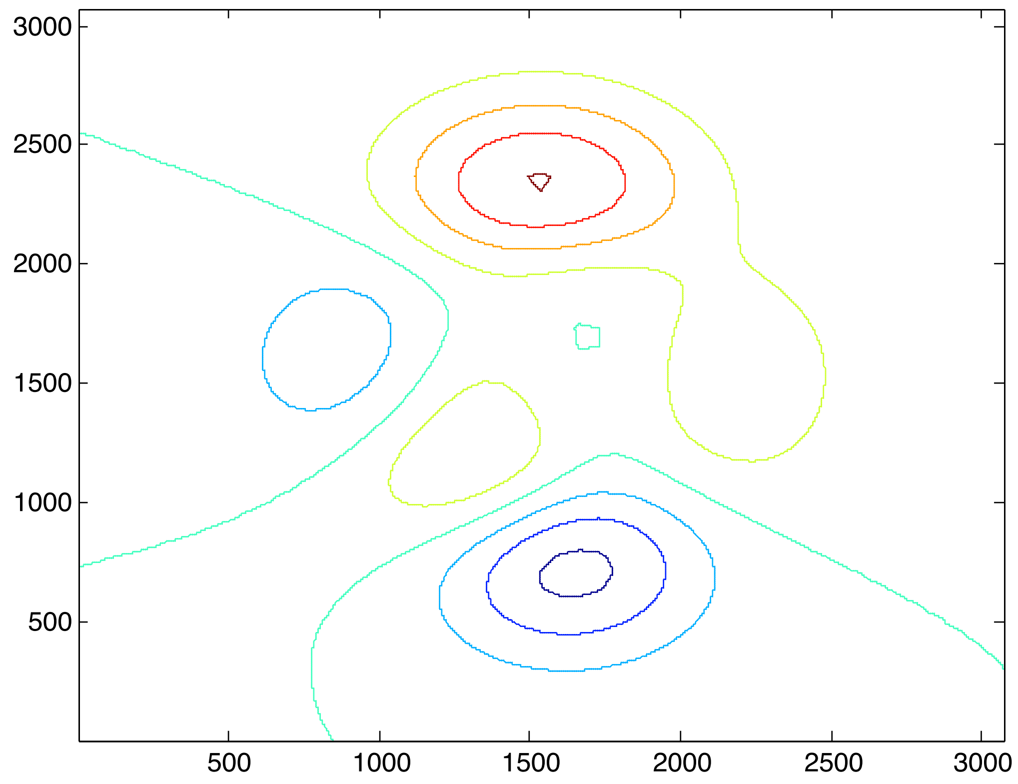
Charles Conley
1933-1984

SIMPLE DYNAMICS

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The associated gradient system is

$$\dot{x} = -\nabla V(x)$$



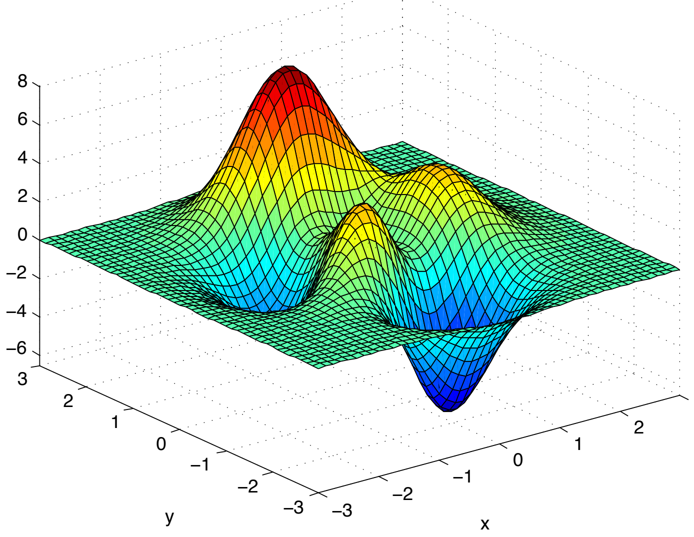


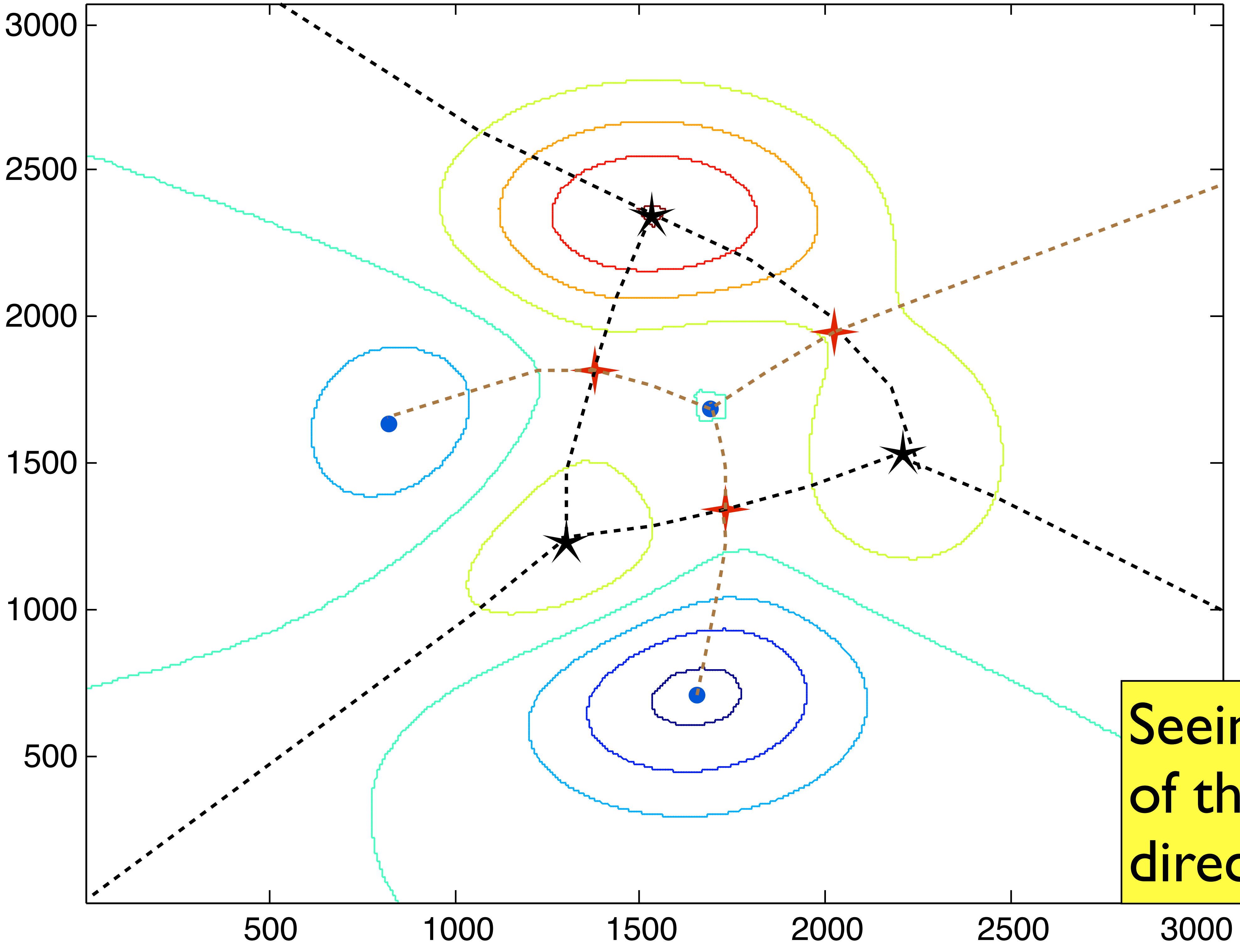
Dynamics is “simple”. Invariant sets made up of

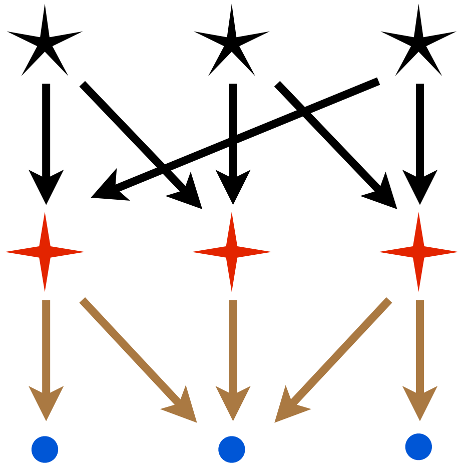
Equilibria

Heteroclinic Orbits

There are special heteroclinic orbits that characterize the qualitative features of the global dynamics.

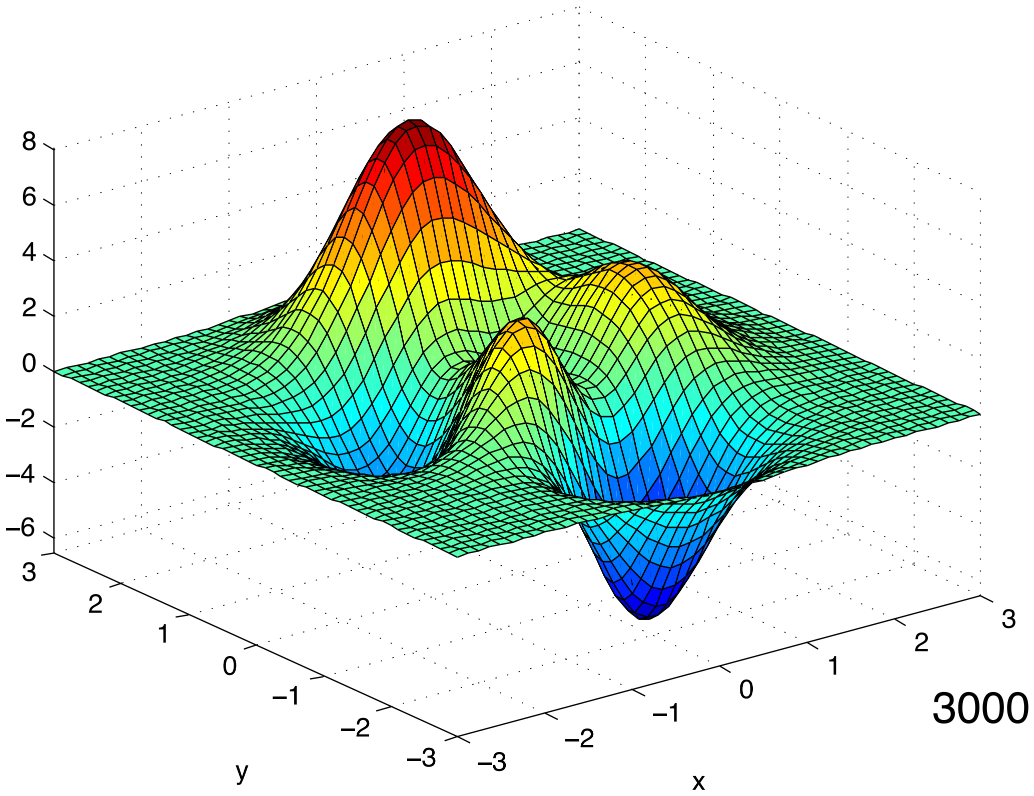


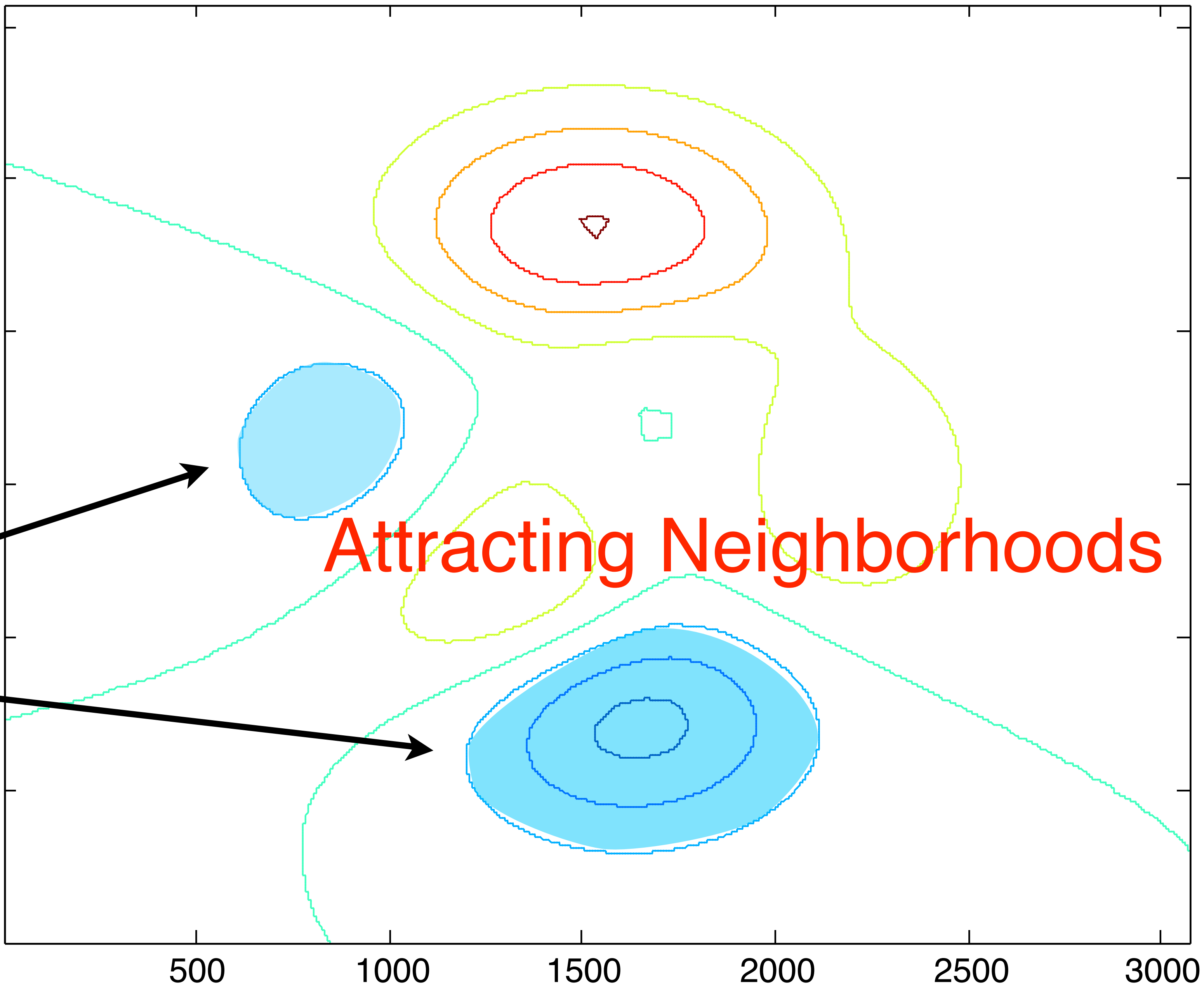




Morse Graph

Seeing the dynamics of these orbits directly is hard!

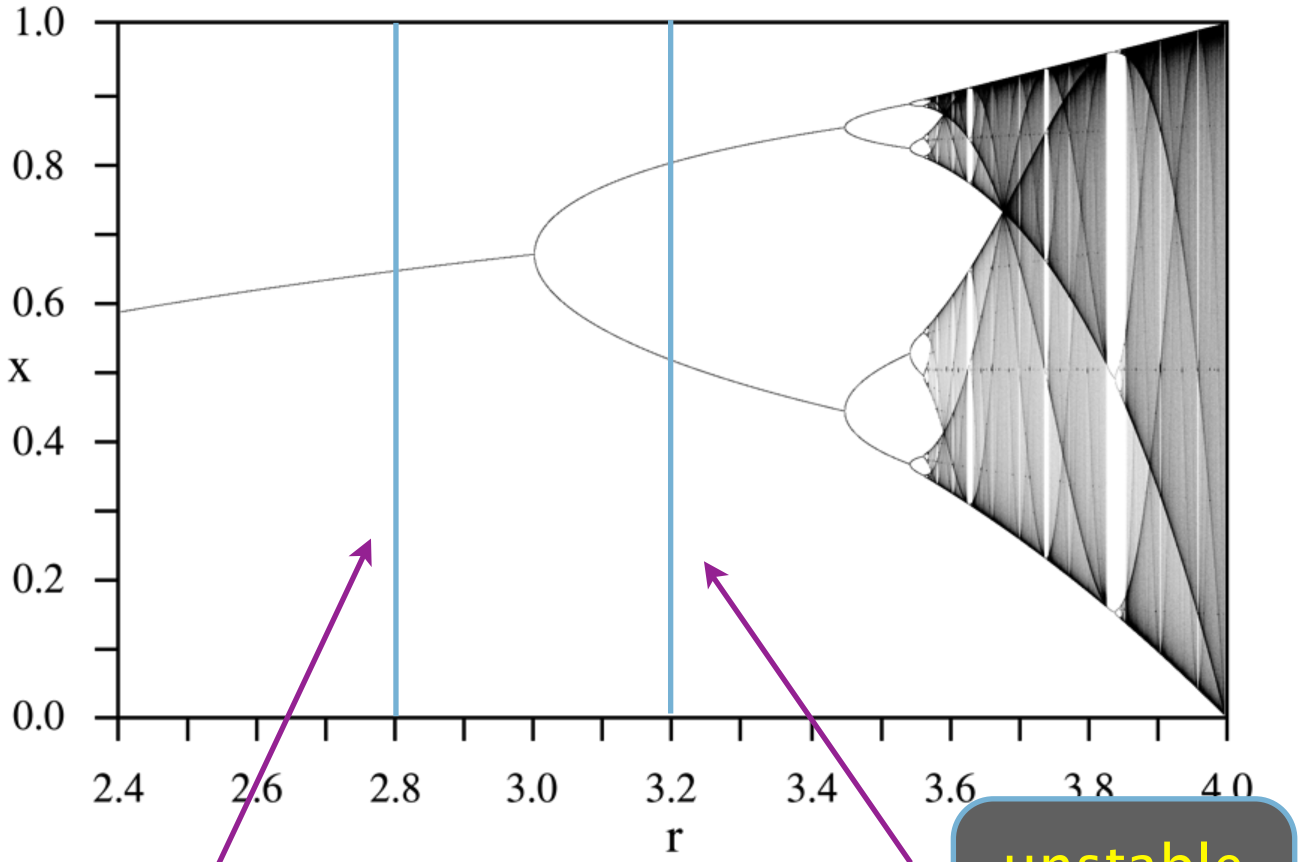




Attracting Neighborhoods

These regions are easily observed from the dynamics.

COMPLICATED DYNAMICS



unstable equilibrium (origin)

non-recurrent dynamics

stable equilibrium

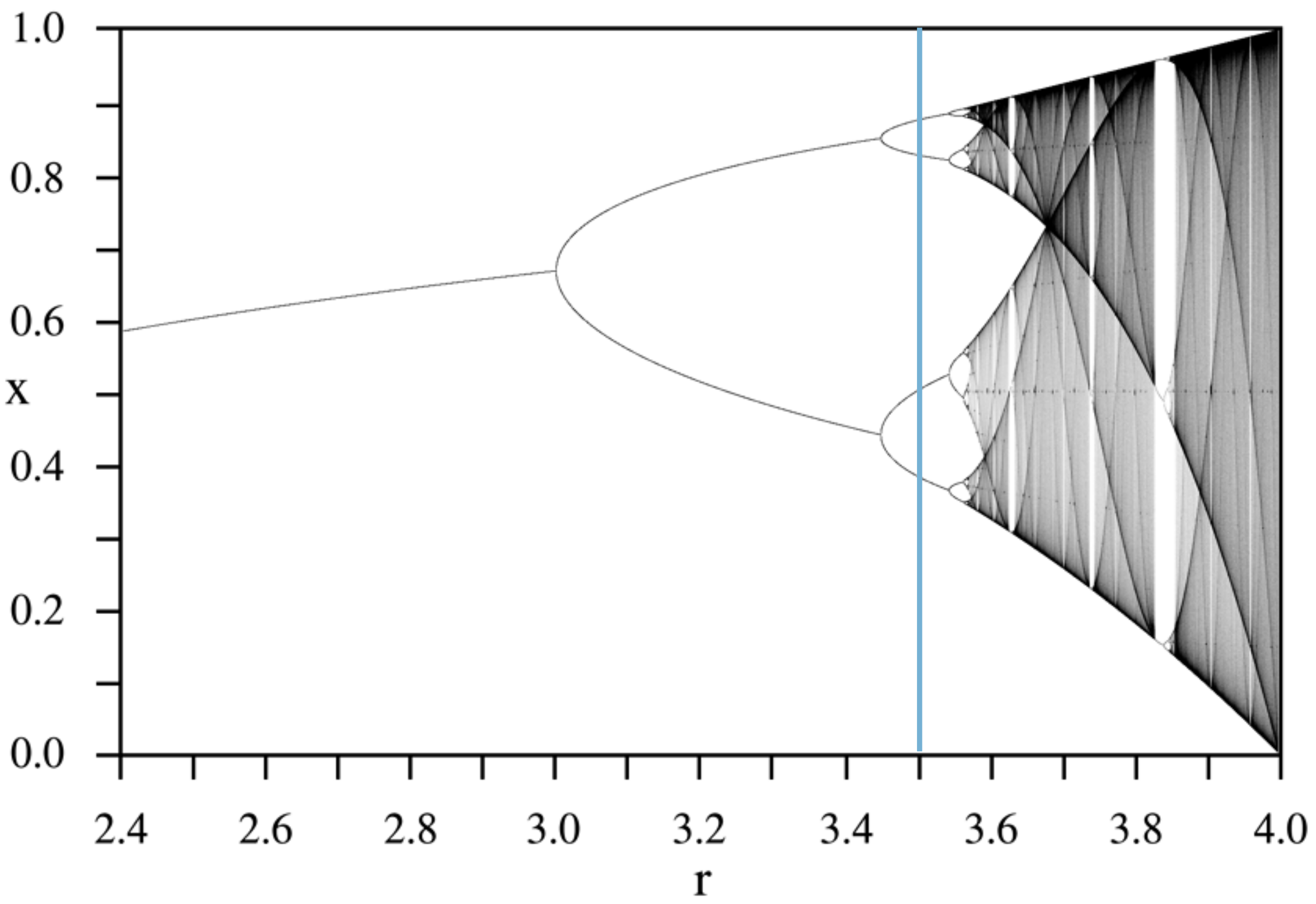
unstable equilibrium (origin)

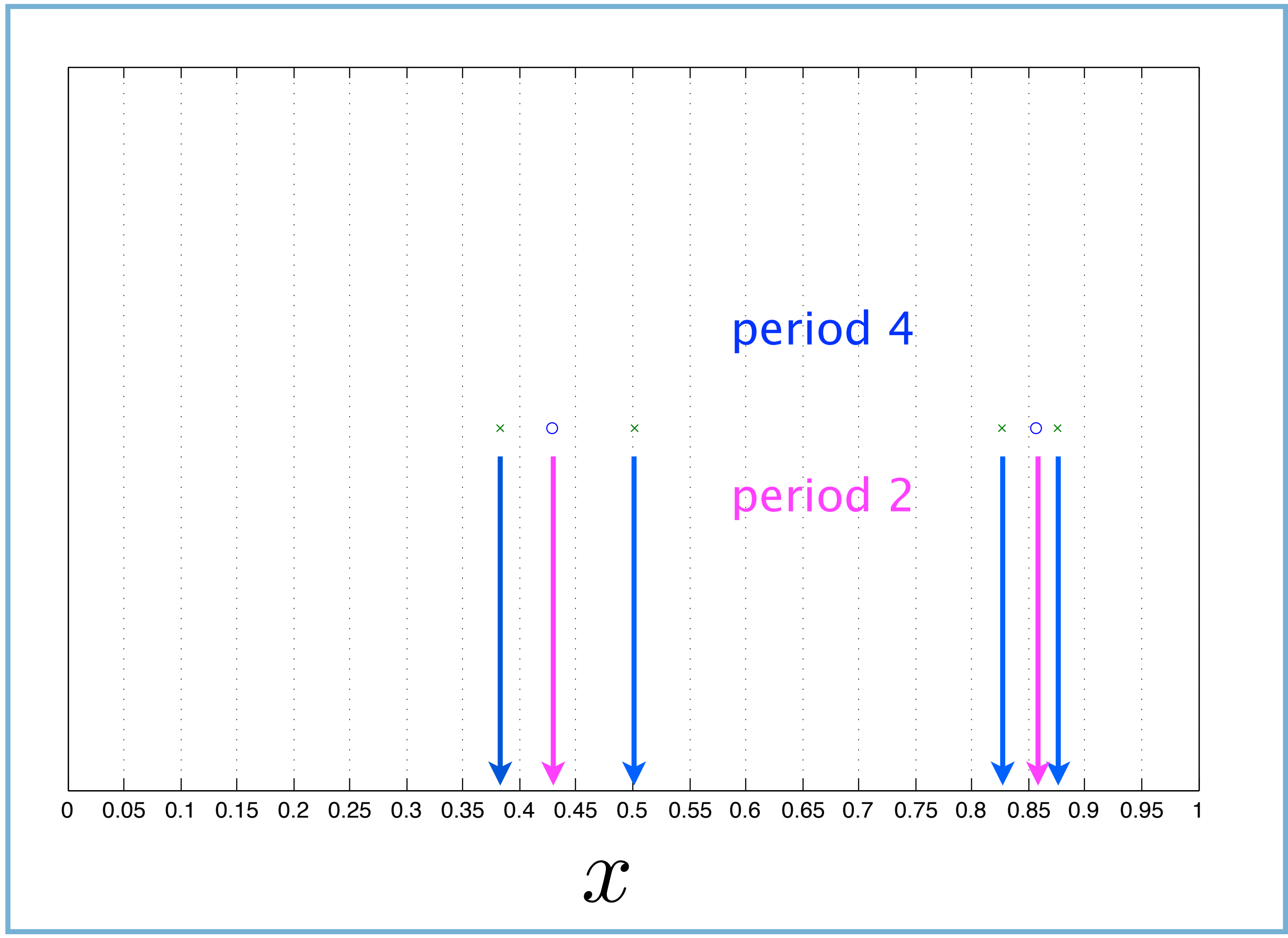
non-recurrent dynamics

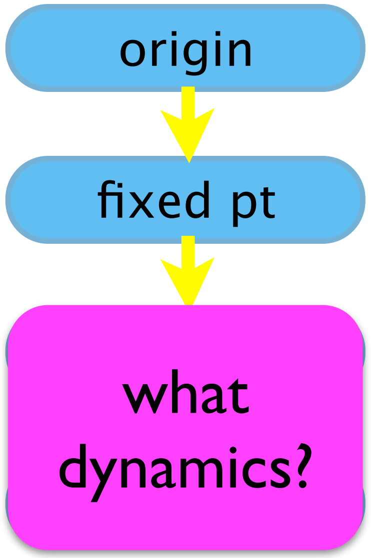
unstable equilibrium

non-recurrent dynamics

stable period 2 orbit







origin

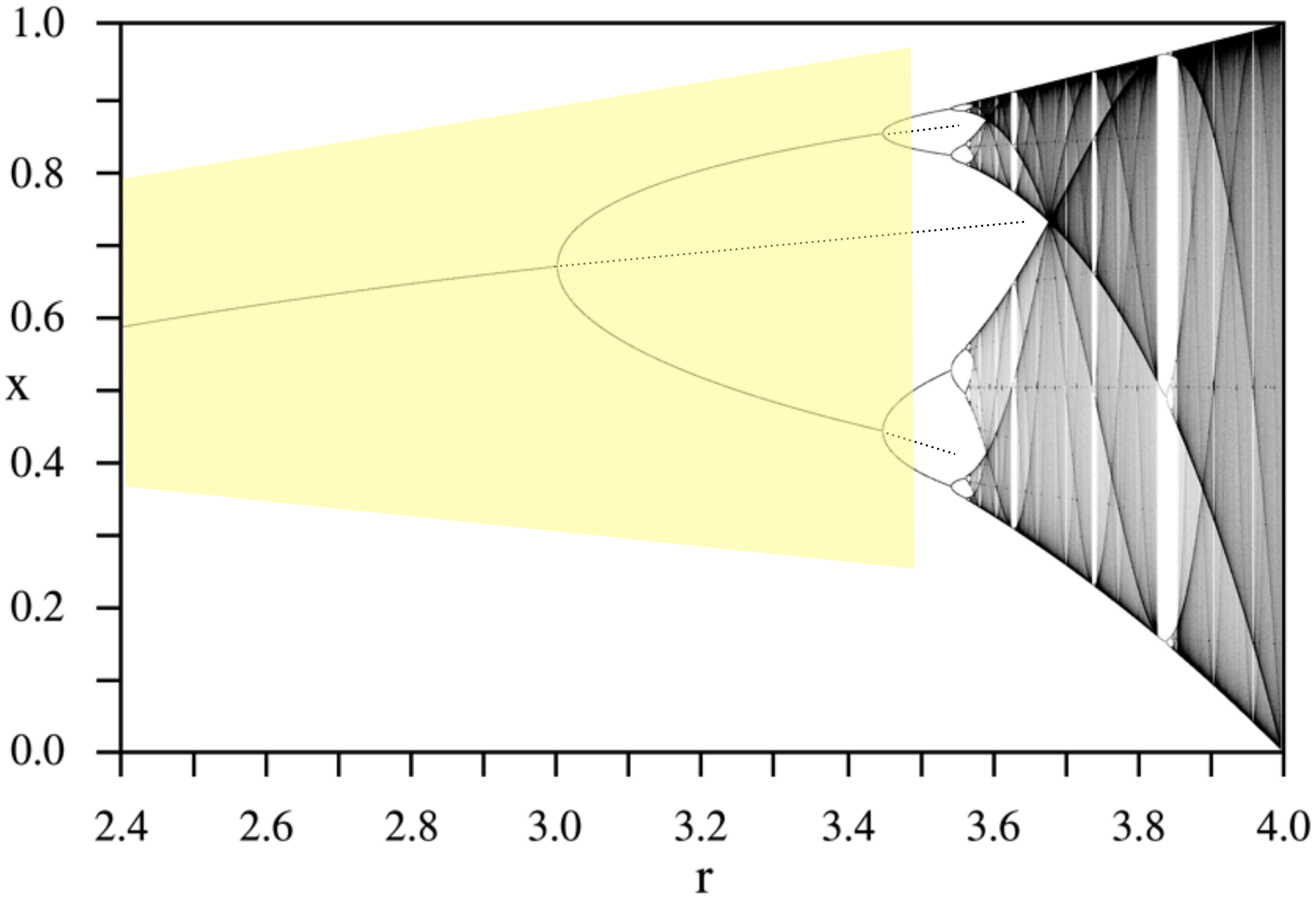
fixed pt

what dynamics?

With fixed precision it is impossible to identify every invariant set

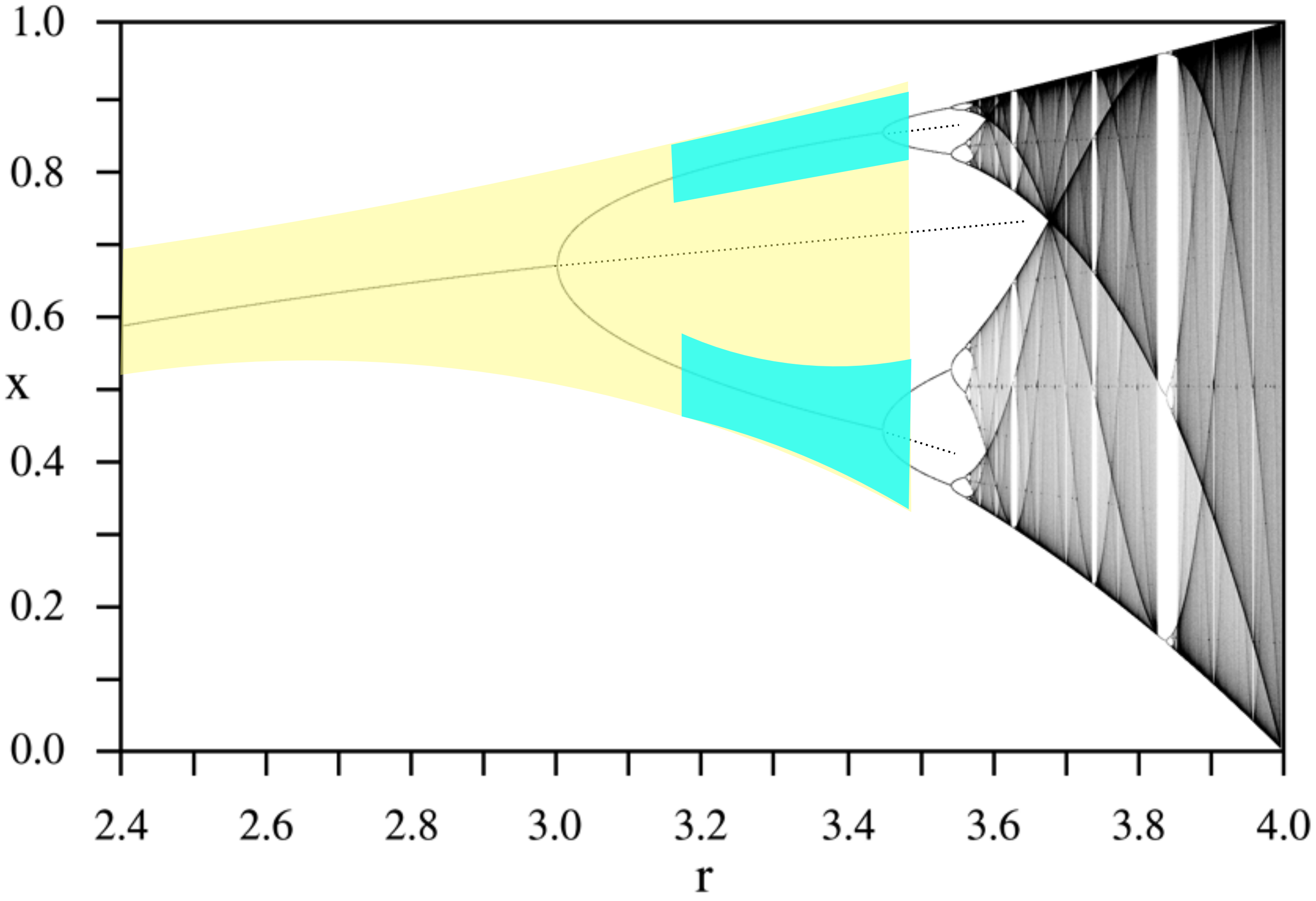
Conley defined the concept of a Morse decomposition to deal with this problem

ATTRACTORS



Coarse Measurements
(large noise)

What type of dynamics
is happening
inside the regions?



Fine Measurements
(little noise)

GOAL

Develop a combinatorial/algebraic topological theory of nonlinear dynamics that is

1. rich enough to capture and identify significant structures of nonlinear dynamics.
2. robust with respect to measurement and modeling imprecisions.
3. computationally tractable.

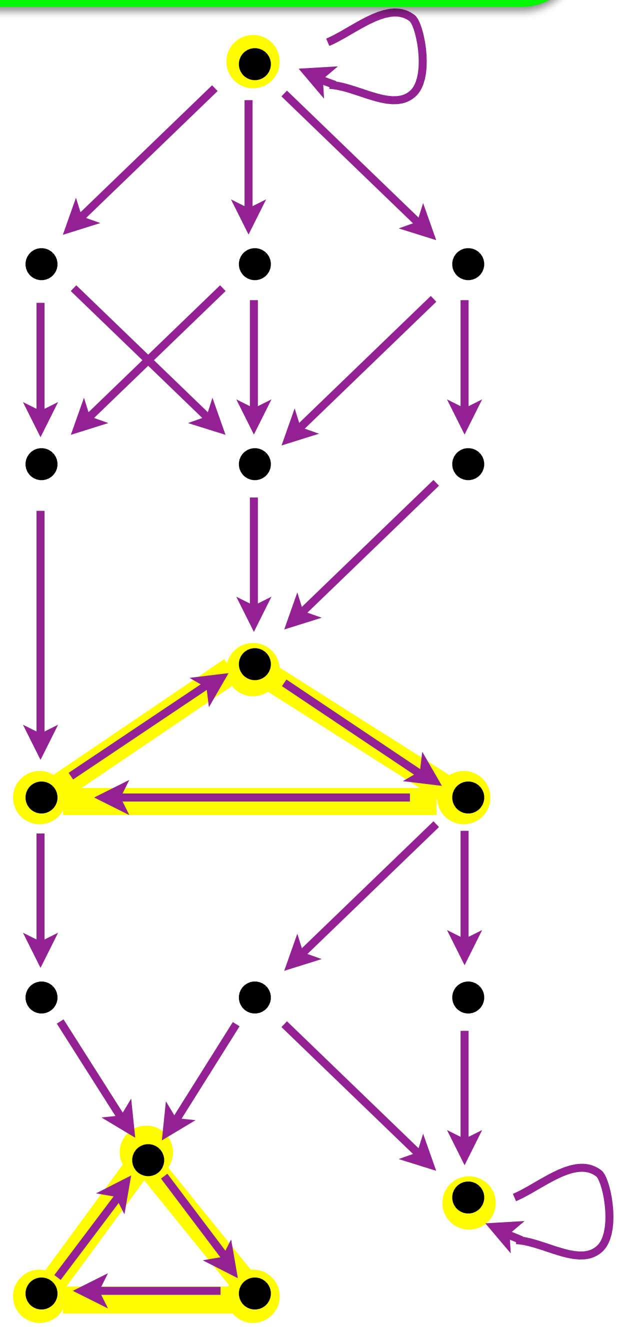
PURELY COMBINATORIAL DYNAMICS

(the information that a computer can process)

Multivalued Map

$\mathcal{F}: \mathcal{X} \rightrightarrows \mathcal{X}$

Directed Graph



Strongly connected path components

Node = States

Directed Edges = Dynamics

Goal: Results should be robust with respect to measurements, noise and modeling.

If we do not know the exact current state, then we cannot know the exact next state.

Remark: For interesting systems the number of states is enormous! We need to simplify.

Basic decomposition of dynamics:

Recurrent

vs.

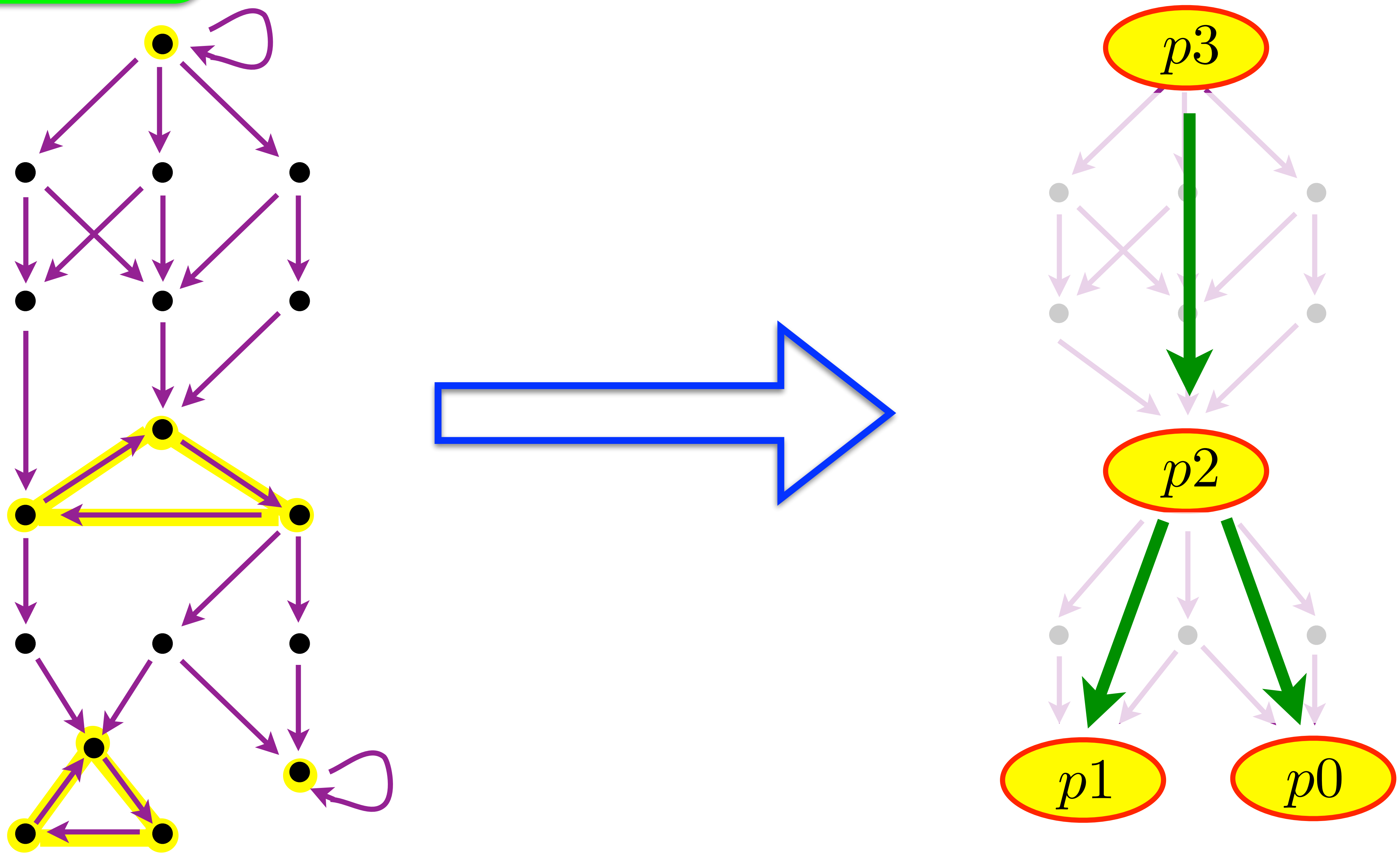
Nonrecurrent

Simplifying the Dynamics

Multivalued Map

$\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$

Directed Graph



Strongly connected path components

POSET

Morse Graph

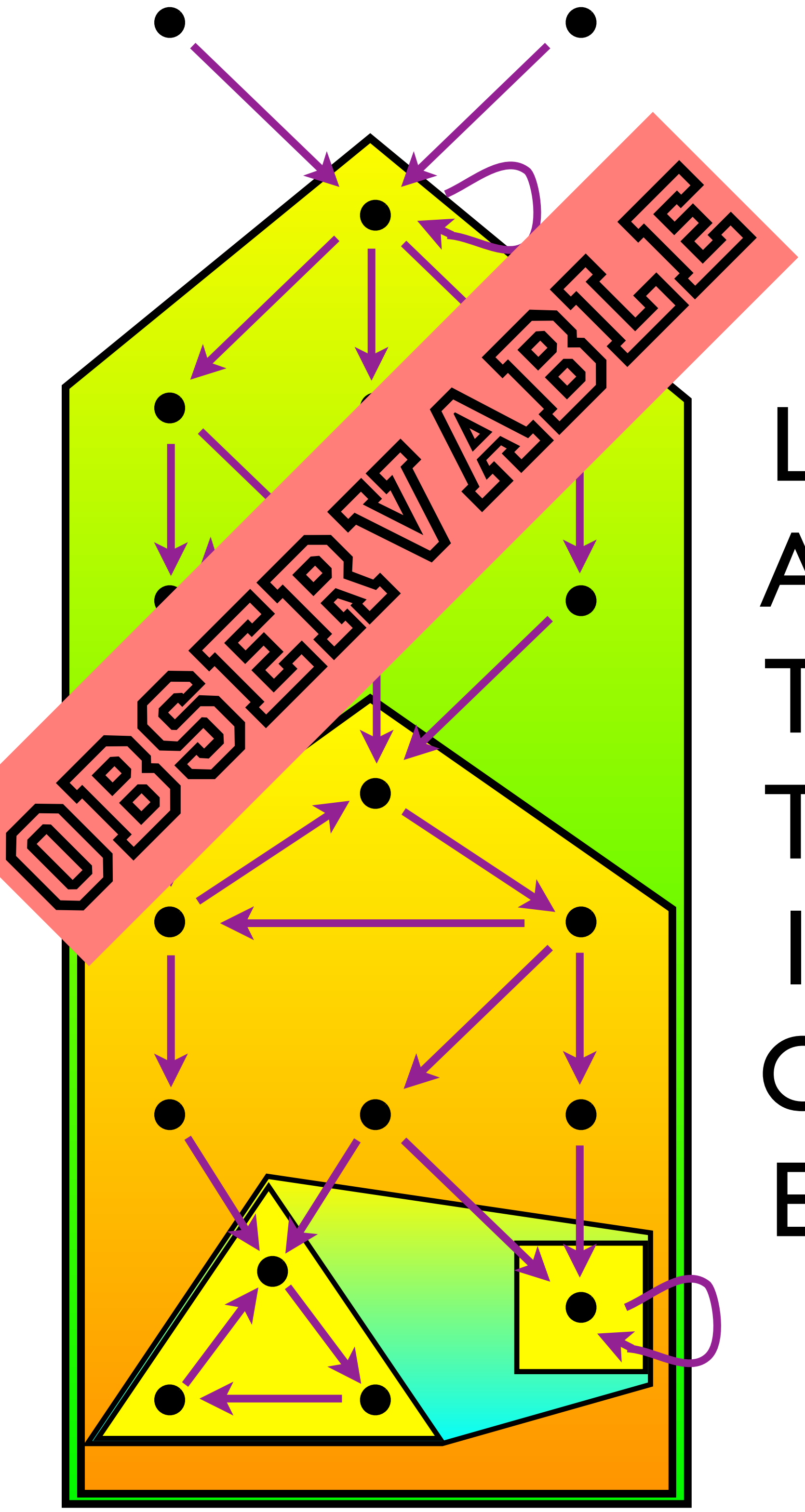
of the Directed Graph

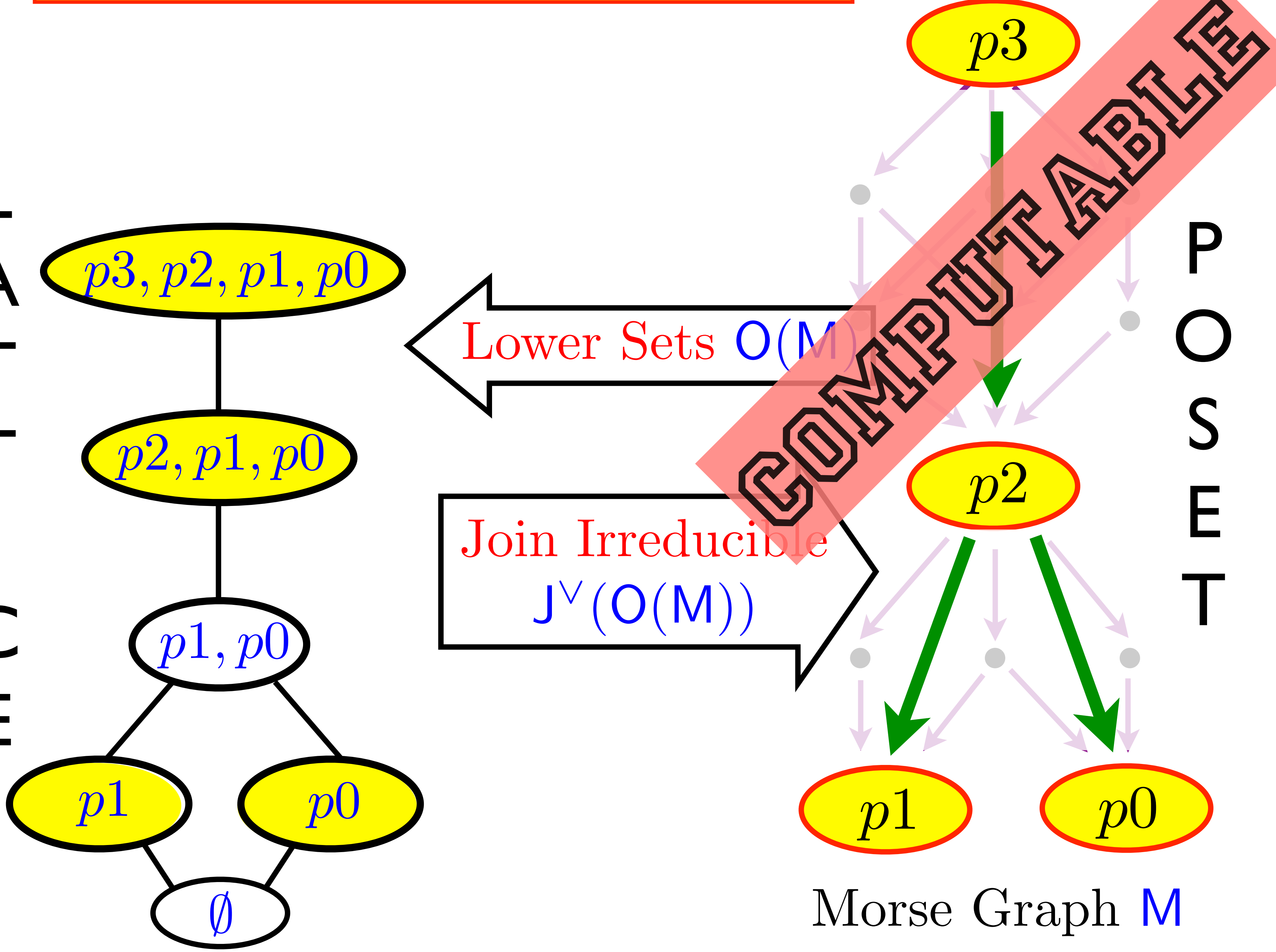
$\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$

What is observable?

$\mathcal{A} \subset \mathcal{X}$ is an attractor if $\mathcal{F}(\mathcal{A}) = \mathcal{A}$

Birkhoff's Theorem implies that the Morse graph and the lattice of Attractors are equivalent.





FROM
CONTINUOUS DYNAMICS
TO
COMBINATORIAL DYNAMICS

(approximation/reconstruction)

NOTATION

Assume: There exists a continuous (deterministic) model for the dynamics

$$f : X \times \Lambda \to X$$

X state space

Λ parameter space

I am not assuming that f is explicitly known!

More natural to view as a Parameterized Dynamical System

$$F : X \times \Lambda \to X \times \Lambda$$

$$F(x, \lambda) = (f_\lambda(x), \lambda) = (f(x, \lambda), \lambda)$$

Given $\Lambda_0 \subset \Lambda$ denote the restriction of F to $X \times \Lambda_0$ by

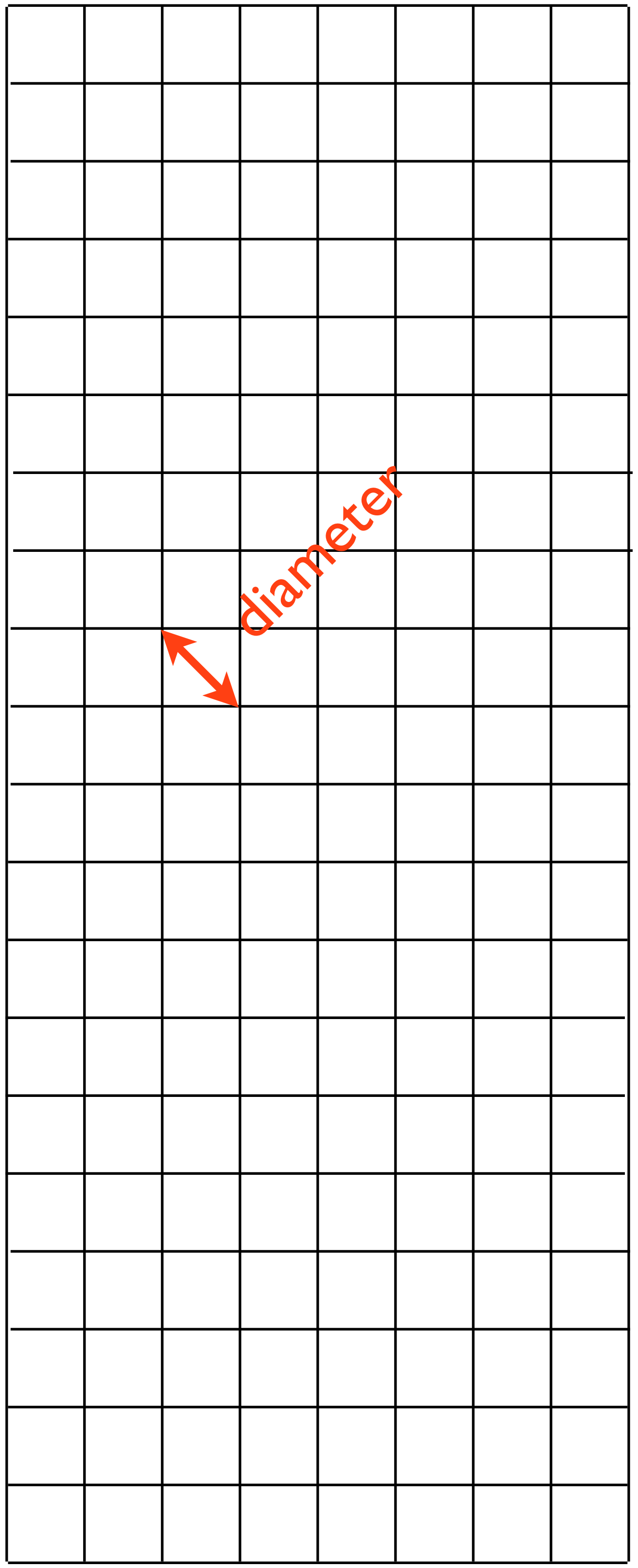
$$F_{\Lambda_0} : X \times \Lambda_0 \to X \times \Lambda_0$$

A grid on a compact metric space X is a finite collection $\mathcal{X}$ of nonempty compact subsets of X satisfying:

1. $X = \bigcup_{\xi \in \mathcal{X}} \xi$
2. $\xi = \mathrm{cl}(\mathrm{int}(\xi))$ for all $\xi \in \mathcal{X}$
3. $\xi \cap \mathrm{int}(\xi') = \emptyset$ for all $\xi \neq \xi' \in \mathcal{X}$

Theorem: For any $\epsilon > 0$ there exists a grid $\mathcal{X}$ of X such that $\mathrm{diam}(\mathcal{X}) < \epsilon$

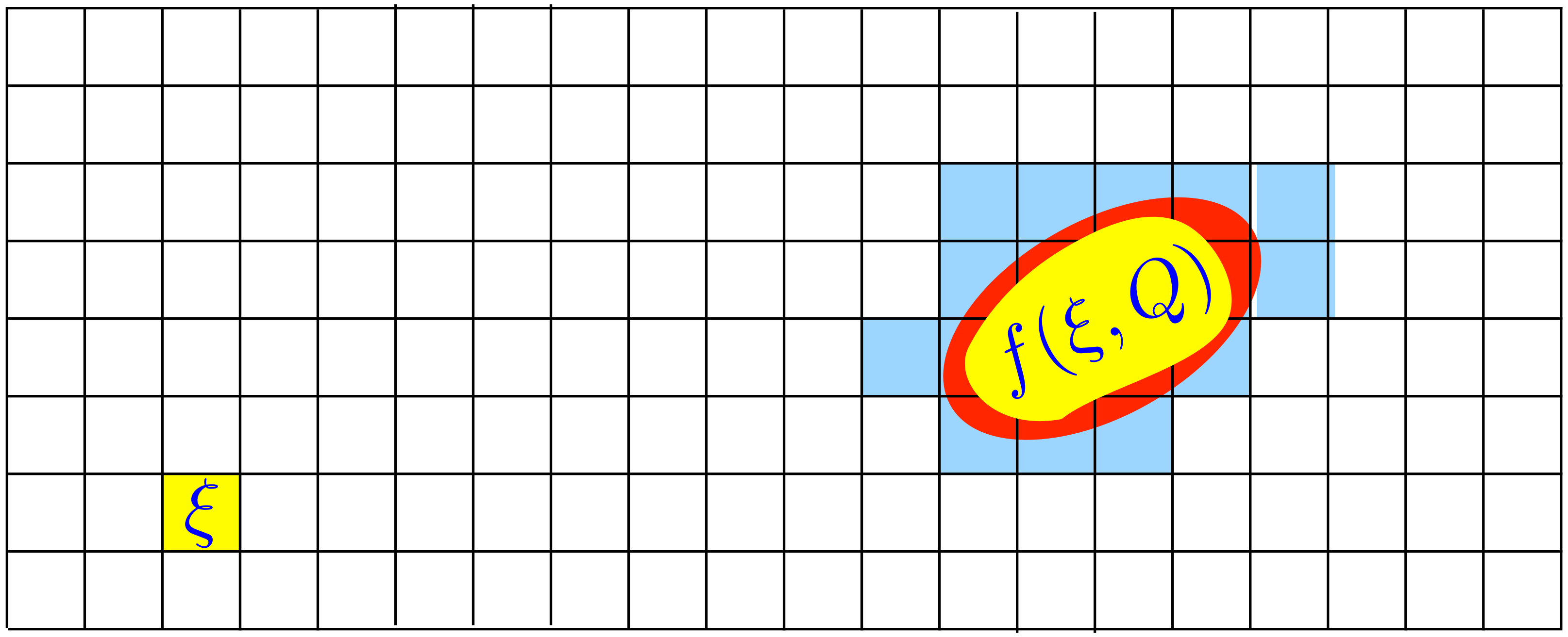
W. Kalies, K.M., R. Vandervorst



A grid in $\mathbb{R}^2$

BUILDING THE MULTIVALUED MAP
(GIVEN f AND $Q \subset \Lambda$)

Grid $\mathcal{X}$ that covers X



$\xi \mapsto B_\delta\left(f(\xi, Q)\right)$ Numerical Error

$$\mathcal{F}_Q(\xi) := \{\xi' \in \mathcal{X} \mid \xi' \cap B_\delta\left(f(\xi, Q)\right) \neq \emptyset\}$$

$\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$ is an outer approximation of f if $f(\xi, Q) \subset \text{int}(|\mathcal{F}_Q(\xi)|)$

BUILDING THE MULTIVALUED MAP
(FROM TIME SERIES)

Proposition: Let $f, g\colon X \to X$ be continuous maps. Let $\mathcal{X}$ be a grid for X. If $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is an outer approximation for f and if g is sufficiently close to f, then $\mathcal{F}$ is an outer approximation for g.

Remark: Assume that I can use $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ to obtain a result about the dynamics of f. The the same result is true for the dynamics of g.

This implies that we have a robust description of the dynamics.

RECOVERING DYNAMICS

(Combinatorial Information)

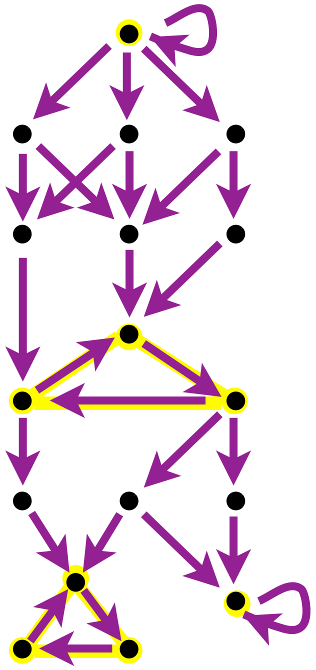
DYNAMICS THAT WE HAVE CAPTURED

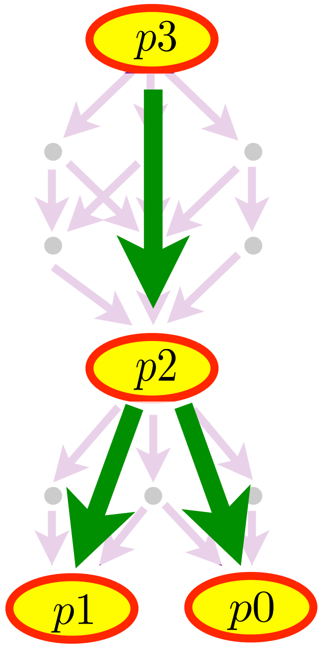
Outer approximation $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ (numerical/data analysis)

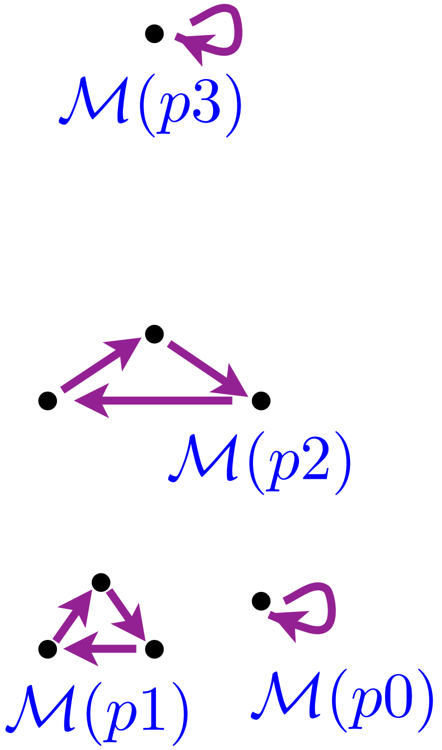
The Morse graph of $\mathcal{F}$ $\mathsf{M} = \{p \mid p \in (\mathsf{P}, \leq)\}$

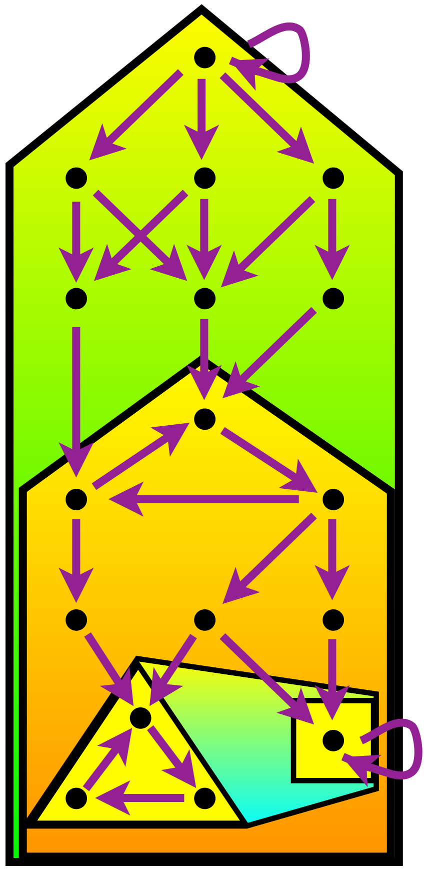
The associated SCPC $\mathcal{M} = \{\mathcal{M}(p) \subset \mathcal{X} \mid p \in (\mathsf{P}, \leq)\}$

The attractor lattice grid elements

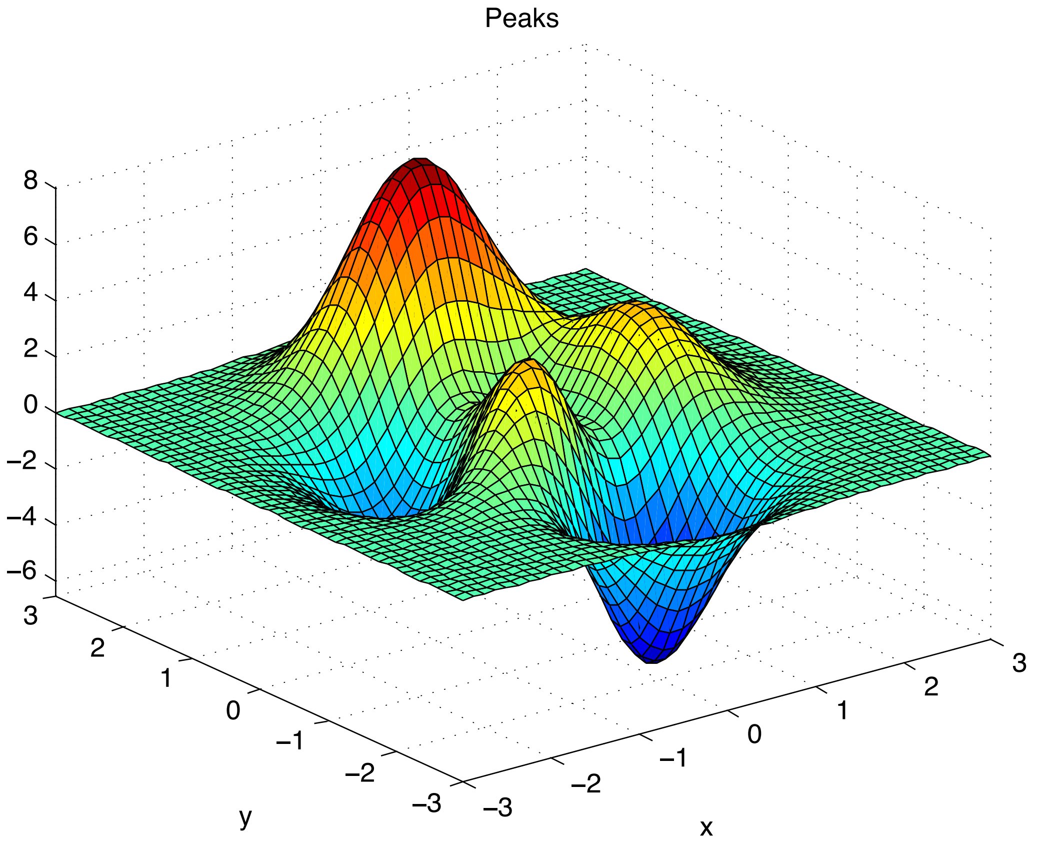


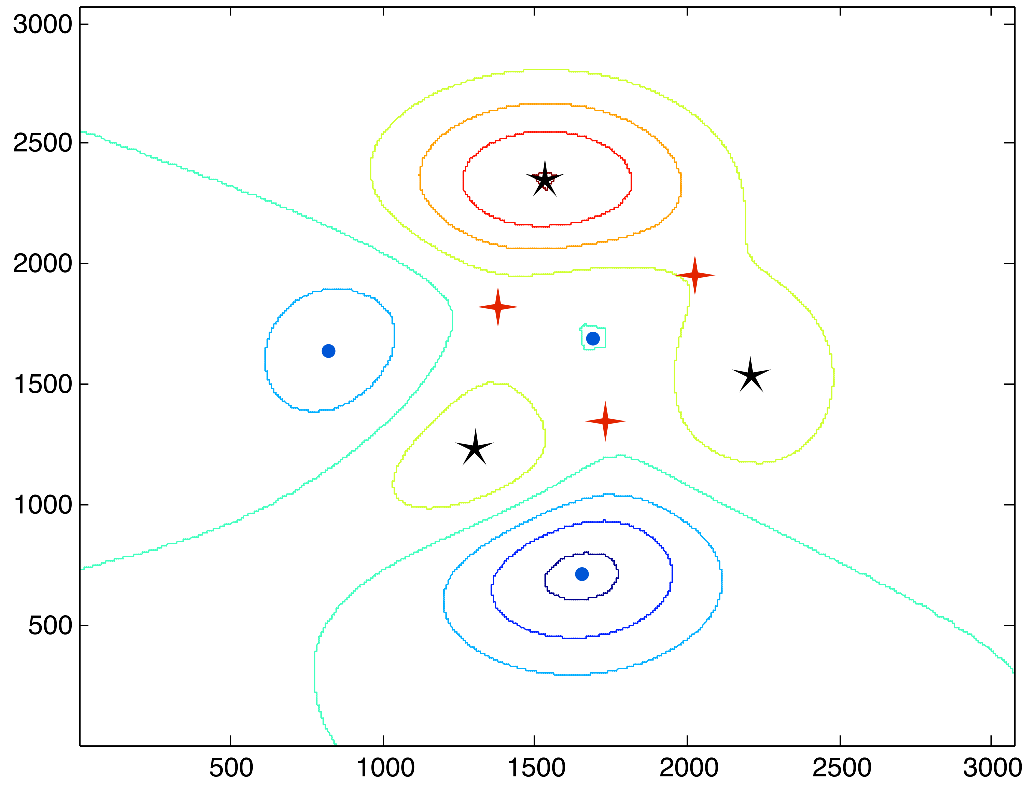


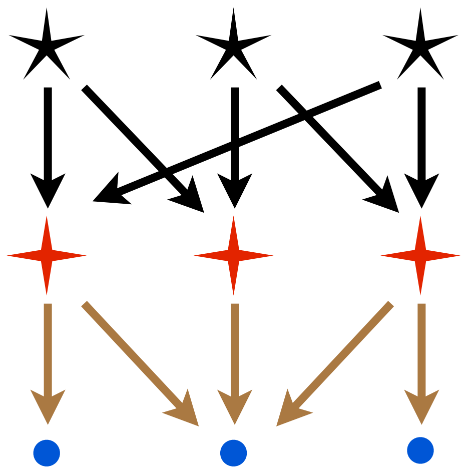




Attractor lattice plus Birkhoff's theorem proves that the dynamics of f cannot go against the directions indicated by the Morse graph.







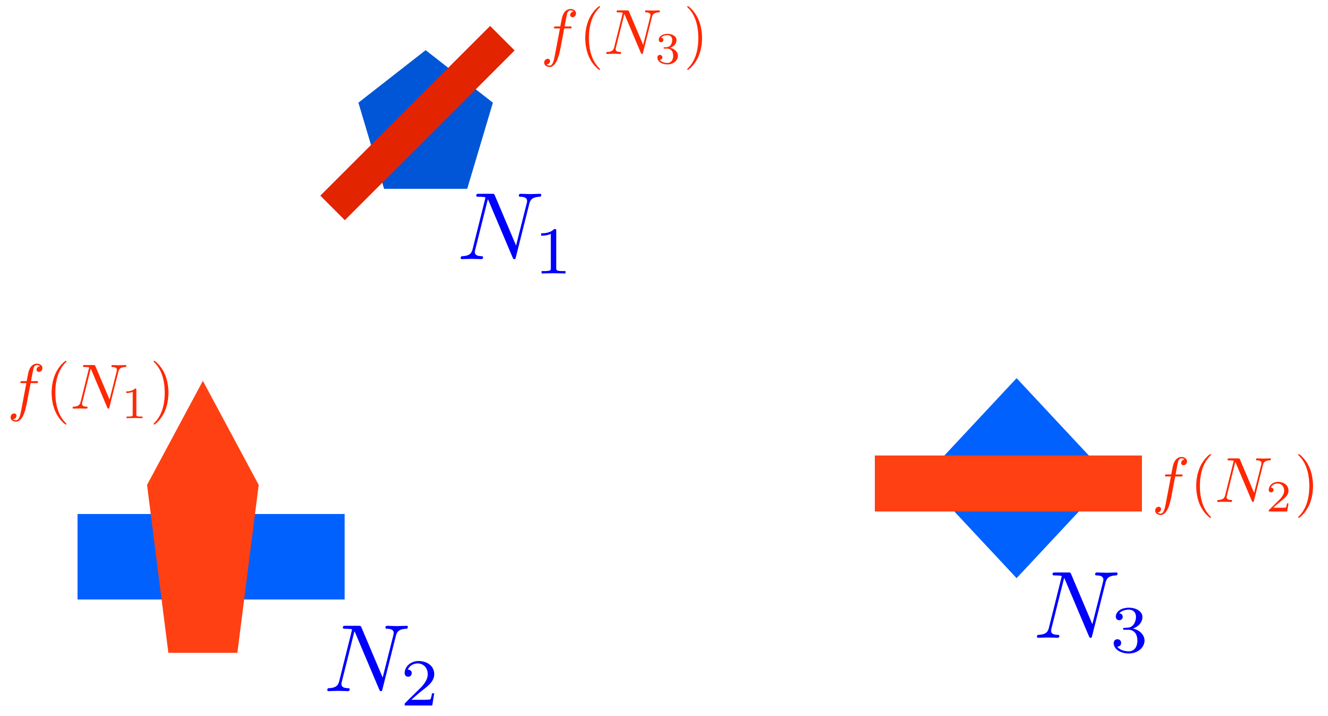
Morse Graph

Warning: we have not shown that there are orbits that correspond to the arrows, but if there is no path in the Morse graph then there is no orbit in the dynamics.

DESCRIBING RECURRENT DYNAMICS

Let $f\colon X \to X$ be continuous. A nonempty isolated invariant set $S \subset X$ is a T-cycle set if there exist T disjoint compact regions $N_1, \ldots, N_T$ such that $N = \cup_{i=1}^{T} N_i$ is an isolating neighborhood for S and

$$f(N_i) \cap N \subset N_{i+1}, \quad i = 0, \ldots, T \quad (N_0 = N_T)$$



S is an attracting T-cycle set if $f(N_i) \subset N_{i+1}$, $i = 0, \ldots, T$

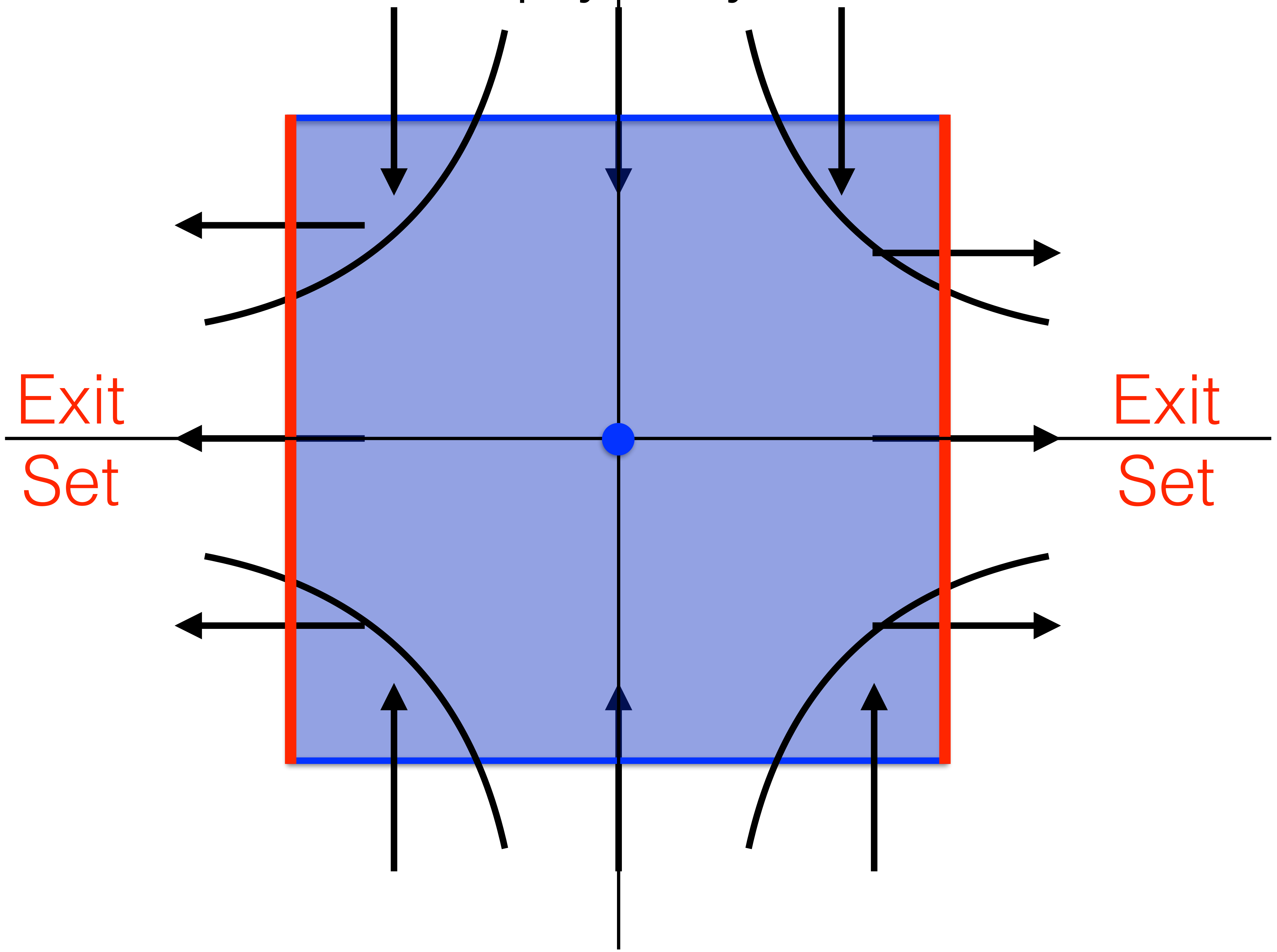
RECOVERING DYNAMICS

(Algebraic Topology)

$\dot{x} = f(x)$

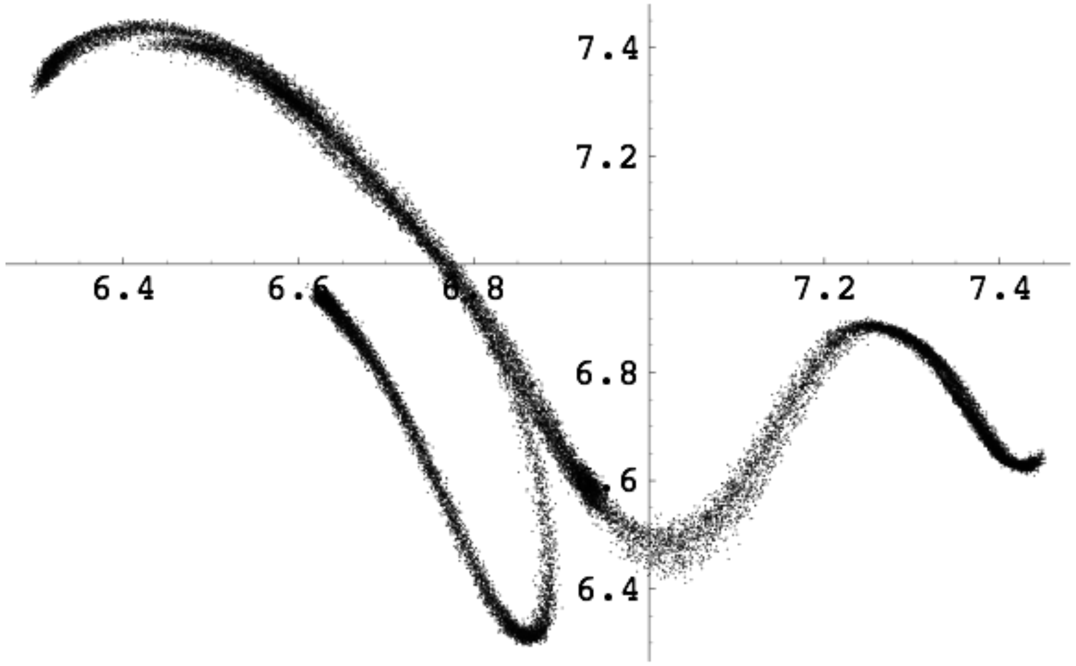
Fixed point (saddle)

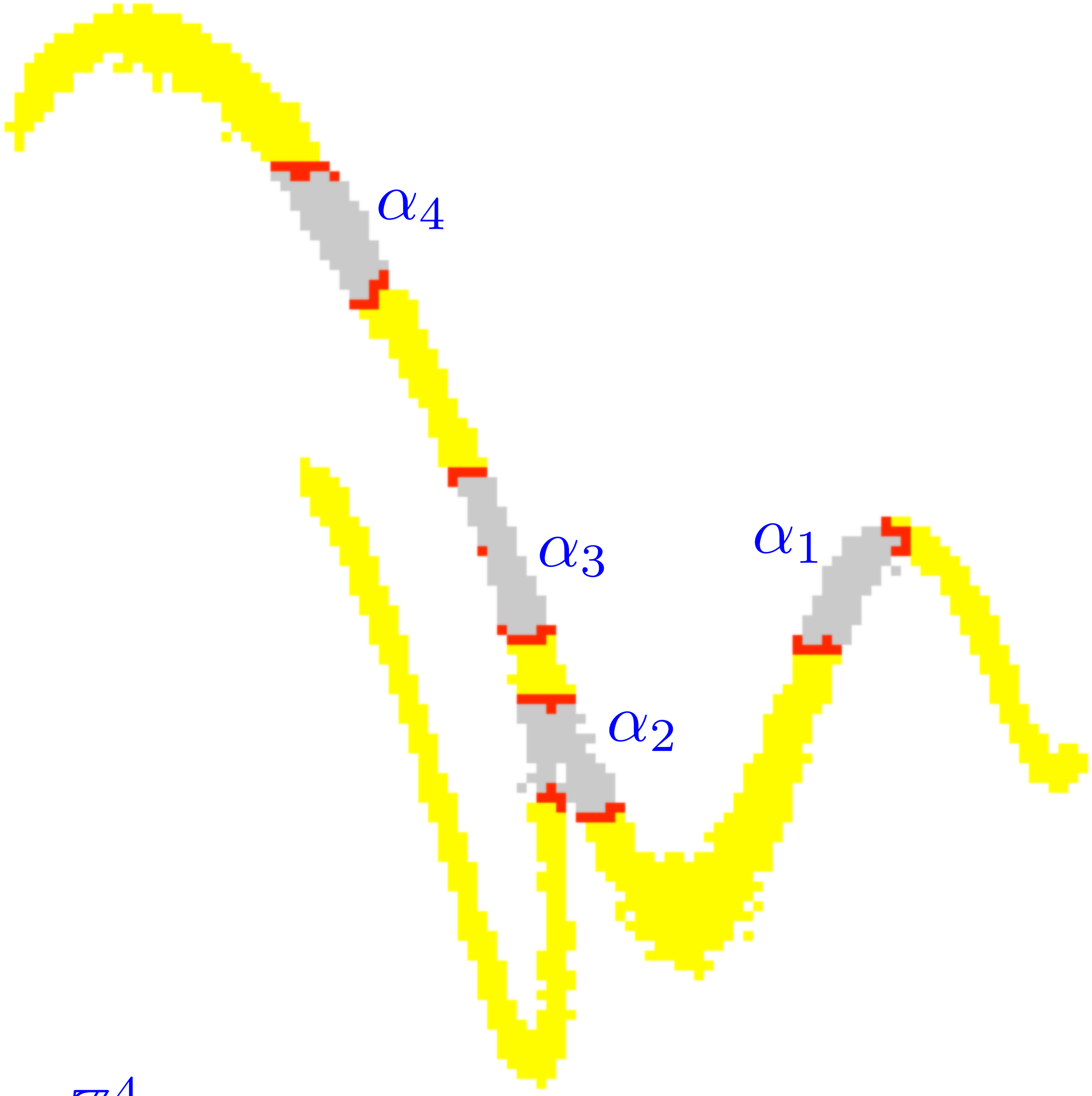
Nonempty 1 cycle set



Topology of neighborhood different from topology of exit set

MAGNETOELASTIC RIBBON





$$f_{P,1} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} : \mathbb{Z}_2^4 \to \mathbb{Z}_2^4$$

K.M., M. Mrozek, J. Reiss, A. Szymczak, PRL (1999)