# 1 Prelimilaries

## 1.1 A review of linear algebra

**Vector Space** Let $I\!R$ be the scalar field of real numbers. We consider only real vector spaces. Let $V_n$ be a set. $V_n$ is a vector space (also called a linear space) if it is equipped with two operations:

$$\text{scalar product} \quad I\!R \times V_n \to V_n,$$
$$\text{vector addition} \quad V_n \times V_n \to V_n,$$

and it is closed under these two operations. That is, $V_n$ is a vector space if $\forall\, \alpha, \beta \in I\!R$ & $\forall\, \mathbf{a}, \mathbf{b} \in V_n$,

$$\alpha\mathbf{a} + \beta\mathbf{b} \in V_n.$$

The vector space $V_n$ is $n$**-dimensional** if we can find a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\} \subset V_n$ such that for any $\mathbf{a} \in V_n$, we have a unique decomposition

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i,$$

where $a_i \in I\!R$ $(i = 1, \cdots n)$ are the components (coordinates) of vector $\mathbf{a}$ under the basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$.
**Tensor Space** Let $V_n$ ($V_m$) be $n$-dimensional ($m$-dimensional) vector space. A mapping $\mathbf{A} : V_n \to V_m$ is a tensor if $\mathbf{A}$ is linear. That is, $\forall\, \alpha, \beta \in I\!R$ & $\forall\, \mathbf{a}, \mathbf{b} \in V_n$,

$$\mathbf{A}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{A}(\mathbf{a}) + \beta\mathbf{A}(\mathbf{b}). \tag{1}$$

Let $\mathrm{Lin}(V_n, V_m)$ be the collection of all linear mappings (i.e., tensors) with domain $V_n$ and range $V_m$. For any $\alpha \in I\!R$ and any $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{Lin}(V_n, V_m)$, define two operations

$$\text{scalar product} \quad (\alpha\mathbf{A}_1)(\mathbf{a}) = \alpha\mathbf{A}_1(\mathbf{a}) \qquad \forall\, \mathbf{a} \in V_n,$$
$$\text{vector addition} \quad (\mathbf{A}_1 + \mathbf{A}_2)(\mathbf{a}) = \mathbf{A}_1(\mathbf{a}) + \mathbf{A}_2(\mathbf{a}) \qquad \forall\, \mathbf{a} \in V_n.$$

♦ CLAIM: For any $\alpha, \beta \in I\!R$ and any $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{Lin}(V_n, V_m)$, $\alpha\mathbf{A}_1 + \beta\mathbf{A}_2$ is a linear mapping (from $V_n$ to $V_m$).

The above claim implies that the set $\mathrm{Lin}(V_n, V_m)$ is also a vector space.
**Inner Product** We equip a $n$-dimensional vector space $V_n$ with a mapping $V_n \times V_n \to I\!R$, called inner product such that for any $\alpha, \beta \in I\!R$ and any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V_n$, the inner product is

1. Positive-definite: $\mathbf{a} \cdot \mathbf{a} \geq 0$; $\mathbf{a} \cdot \mathbf{a} = 0 \iff \mathbf{a} = 0$,
2. Linear: $\mathbf{a} \cdot (\alpha\mathbf{b} + \beta\mathbf{c}) = \alpha\mathbf{a} \cdot \mathbf{b} + \beta\mathbf{a} \cdot \mathbf{c}$,
3. Symmetric: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

Geometric interpretations:

- Length of a vector: $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$,
- Angle between two vectors: $\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

**Euclidean Space $I\!R^n$** For a $n$-dimensional vector space $V_n$ equipped with an inner product, we can find an orthonormal basis $\{\mathbf{e}_i : i = 1, \cdots, n\}$ such that for all $i, j = 1, \cdots, n$,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\delta_{ij}$ is called *kronecker delta*. With respect to this basis, for any vector $\mathbf{a} \in V_n$, we find its components $(a_1, \cdots, a_n)$ (or coordinates if $a$ is a point in space)

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i, \qquad a_i = \mathbf{a} \cdot \mathbf{e}_i \in I\!R \quad \forall\, i = 1, \cdots, n.$$

We can further identify the space $V_n$ with the familiar Euclidean space $I\!R^n$. However, one shall keep in mind, $I\!R^n$, as a vector space equipped with an inner product, is more than a collection of arrays of real numbers. One should not think of a vector in $I\!R^n$ as an array of real numbers unless we specify a basis or a frame.

**Tensor Product** For vectors $\mathbf{a} \in I\!R^n$ and $\mathbf{b} \in I\!R^m$, the tensor product $\mathbf{b} \otimes \mathbf{a}$ is a linear mapping:

$$\begin{aligned} &\mathbf{b} \otimes \mathbf{a} : V_n \to V_m \\ &(\mathbf{b} \otimes \mathbf{a})(\mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} \quad \forall\, \mathbf{c} \in I\!R^n. \end{aligned}$$

♦ CLAIM: For any $\mathbf{a} \in I\!R^n$ and $\mathbf{b} \in I\!R^m$, the mapping $\mathbf{b} \otimes \mathbf{a}$ (from $V_n$ to $V_m$) defined above is linear.

♦ CLAIM: Let $\{\mathbf{e}_i : i = 1, \cdots n\}$ be an orthonormal basis of $I\!R^n$ and $\{\hat{\mathbf{e}}_p : p = 1, \cdots m\}$ be an orthonormal basis of $I\!R^m$. Show that

$$\{\hat{\mathbf{e}}_p \otimes \mathbf{e}_i : i = 1, \cdots, n,\ p = 1, \cdots m\} \subset \mathrm{Lin}(I\!R^n, I\!R^m)$$

forms a basis of the linear space $\mathrm{Lin}(I\!R^n, I\!R^m)$.

**Subspace of $I\!R^n$, Orthogonal Subspace** A subset $M \subset I\!R^n$ is a subspace if $\forall\, \alpha, \beta \in I\!R$ & $\forall\, \mathbf{a}, \mathbf{b} \in M$,

$$\alpha \mathbf{a} + \beta \mathbf{b} \in M.$$

Let $M^\perp = \{\mathbf{b} : \mathbf{b} \cdot \mathbf{a} = 0\ \forall\ \mathbf{a} \in M\}$.

♦ CLAIM: Show that $M^\perp$ is a subspace of $I\!R^n$ if $M$ is a subspace.

**Projection Theorem** Let $M$ be a subspace of $I\!R^n$. For any $\mathbf{x} \in I\!R^n$, we have

$$\mathbf{x} = \mathbf{y} + \mathbf{z} \text{ where } \mathbf{y} \in M,\ \mathbf{z} \in M^\perp.$$

The vector $\mathbf{y}$, $\mathbf{z}$ are uniquely determined by $\mathbf{x}$.

♦ PROOF:

**Transpose of a Tensor** Let $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^m)$, $\{\mathbf{e}_i : i = 1, \cdots n\}$ be an orthonormal basis of $I\!R^n$ and $\{\hat{e}_p : p = 1, \cdots m\}$ be an orthonormal basis of $I\!R^m$. Then $\mathbf{A}$ admits the following decomposition

$$\mathbf{A} = \sum_{p,i} A_{pi}\hat{\mathbf{e}}_p \otimes \mathbf{e}_i \qquad \text{where } \; A_{pi} = \hat{\mathbf{e}}_p \cdot \mathbf{A}(\mathbf{e}_i) \;\; \forall\, i = 1, \cdots, n, p = 1, \cdots, m.$$

Define

$$\begin{aligned} &\mathbf{A}^T : I\!R_m \to I\!R^n, \\ &\mathbf{A}^T = \sum_{p,i} A_{pi}\mathbf{e}_i \otimes \hat{\mathbf{e}}_p \in \mathrm{Lin}(I\!R^m, I\!R^n). \end{aligned}$$

♦ CLAIM: For any $\mathbf{a} \in I\!R^n$ and $\mathbf{b} \in I\!R^m$,

$$\mathbf{b} \cdot \mathbf{A}(\mathbf{a}) = \mathbf{a} \cdot \mathbf{A}^T(\mathbf{b}).$$

**Symmetric and Skew-symmetric Tensor** Let $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^n)$. $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^T$; $\mathbf{A}$ is skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.

Let $\{\mathbf{e}_i : i = 1, \cdots n\}$, $\{\hat{\mathbf{e}}_p : p = 1, \cdots n\}$ be two orthonormal bases of $I\!R^n$. We have shown

$$\mathbf{A} = \sum_{p,i} A_{pi}\hat{\mathbf{e}}_p \otimes \mathbf{e}_i \qquad \text{where } \; A_{pi} = \hat{\mathbf{e}}_p \cdot \mathbf{A}(\mathbf{e}_i) \;\; \forall\, p, i = 1, \cdots, n.$$

♦ CLAIMS :

1. For any $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^n)$, we have a unique decomposition $\mathbf{A} = \mathbf{E} + \mathbf{W}$, where $\mathbf{E} = \mathbf{E}^T$ and $\mathbf{W} = -\mathbf{W}^T$.
2. $\mathbf{A} = \mathbf{A}^T$ if and only if for any $\mathbf{a}, \mathbf{b} \in I\!R^n$,
$$\mathbf{b} \cdot \mathbf{A}(\mathbf{a}) = \mathbf{a} \cdot \mathbf{A}(\mathbf{b}).$$
3. If $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{a} \cdot \mathbf{A}(\mathbf{a}) = 0$ for any $\mathbf{a} \in I\!R^n$, then $\mathbf{A} = 0$.
4. There exists a nonzero tensor $\mathbf{A}$ such that
$$\mathbf{a} \cdot \mathbf{A}\mathbf{a} = 0 \qquad \forall\, \mathbf{a} \in I\!R^n,\; n \geq 2.$$
5. Assume that $(\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n) = (\mathbf{e}_1, \cdots, \mathbf{e}_n)$. If $\mathbf{A} = \mathbf{A}^T$, then $A_{pi} = A_{ip}$ for all $p, i = 1, \cdots, n$; if $\mathbf{A} = -\mathbf{A}^T$, then $A_{pi} = -A_{ip}$.

**Product of tensors** Let $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^m)$, $\mathbf{B} \in \mathrm{Lin}(I\!R^m, I\!R^k)$. Then

$$\begin{aligned} &\mathbf{BA} : I\!R^n \to I\!R^k, \\ &\mathbf{BA}(\mathbf{a}) = \mathbf{B}(\mathbf{A}(\mathbf{a})). \end{aligned}$$

**Orthogonal Tensor** Let $\mathbf{Q} \in \mathrm{Lin}(I\!R^n, I\!R^n)$. The tensor $\mathbf{Q}$ is orthogonal if $\mathbf{Qa} \cdot \mathbf{Qb} = \mathbf{a} \cdot \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in I\!R^n$. From the definition we see that orthogonal tensor operating on vectors preserves the length of a vector and the angle between two vectors since

1. $|\mathbf{a}| = |\mathbf{Q}\mathbf{a}|$, and
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{Q}\mathbf{a} \cdot \mathbf{Q}\mathbf{b}$.

♦ CLAIM: A tensor $\mathbf{Q} : I\!R^n \to I\!R^n$ is orthogonal if and only if $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$, where $\mathbf{I}$ is the identity mapping from $I\!R^n$ to $I\!R^n$.

**Trace and determinant of a tensor** Let $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^n)$ and $\{\mathbf{e}_i : i = 1, \cdots, n\}$ be an orthonormal basis. Then we have $\mathbf{A} = \sum_{p,i} A_{pi}\mathbf{e}_p \otimes \mathbf{e}_i$ and refer to $\mathrm{Tr}(\mathbf{A}) = \sum_{p=1}^n A_{pp}$ as the trace of $\mathbf{A}$, $\det A = \det[A_{pi}]$ as the determinant of $\mathbf{A}$.

♦ CLAIM Tr, det : $\mathrm{Lin}(I\!R^n, I\!R^n) \to I\!R$ is independent of the choice of basis.

**Rigid Rotation Tensor** An orthogonal tensor $\mathbf{R} \in \mathrm{Lin}(I\!R^n, I\!R^n)$ is a rigid rotation if $\det \mathbf{R} = +1$.
**Representation theorem**: For any $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R)$, there is an $\mathbf{a} \in I\!R^n$ such that Explicitly, if we have

$$\mathbf{A} = \sum_i A_{1i}\hat{\mathbf{e}}_1 \otimes \mathbf{e}_i, \quad \hat{\mathbf{e}}_1 = 1,$$

then

$$\mathbf{a} = \sum_{i=1}^n A_{1i}\mathbf{e}_i.$$

**Cross product in $I\!R^3$** For $\mathbf{a},\ \mathbf{b} \in I\!R^3$,

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{W}(\mathbf{b}),$$

where $\mathbf{W} = \sum_{p,i} W_{pi}\mathbf{e}_p \otimes \mathbf{e}_i$,

$$[W_{p,i}] = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

♦ CLAIM: The following properties of cross products holds:

1. $\mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b}, \quad \mathbf{a} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0,\ \mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0$.
2. $(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \wedge \mathbf{a}) \cdot \mathbf{b}$.
3. Geometric interpretation: show that $|\mathbf{a} \wedge \mathbf{b}|$ =area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$; $|\mathbf{c} \cdot (\mathbf{a} \wedge \mathbf{b})|$= volume of the parallelepiped formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.